

Charge-Stripe Phases Versus a Weak Anisotropy of Nearest-Neighbor Hopping

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Recently, we demonstrated rigorously the stability of charge-stripe phases in 2D quantum-particle systems that are described by extended Falicov-Kimball Hamiltonians, with the quantum hopping particles being either spinless fermions or hardcore bosons. In this paper, by means of the same methods, we show that an arbitrary small anisotropy of nearest-neighbor hopping eliminates the $\pi/2$ -rotation degeneracy of dimeric and axial-stripe phases and orients them in the direction of the weaker hopping. Due to the same anisotropy the obtained phase diagrams of fermions show a tendency to become similar to those of hardcore bosons.

KEY WORDS: fermion and hardcore boson lattice systems, ground-state phase diagrams, Falicov-Kimball model, charge-stripe phases, hopping anisotropy

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1. INTRODUCTION

In our recent paper⁽¹⁾ we addressed the problem of formation of static charge-stripe phases in systems of correlated quantum particles. Since an experimental evidence for the existence of striped phases in materials exhibiting high-temperature superconductivity has been gathered,^(2,3) one observes vigorous discussions of the problem of the stability of such phases in simple models of correlated electrons. Mainly, some Hubbard-like or $t - J$ -like models are considered in this context (see a review paper by Oleś Ref. 4). Unfortunately, none of the methods applied for investigating these models enables one to control tiny energy differences between competing phases: the phase energies are calculated by means

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of uncontrolled approximations, moreover the results are biased by finite-size and boundary effects. In this situation, to get a deeper insight into the stability problem of striped phases, some simpler analogue problems^(5–8) have been studied.

Adopting such an approach, we have proposed in Ref. 1 a kind of an extended Falicov–Kimball model, whose ground-state phase diagram at the half-filling and at a sufficiently strong coupling is amenable to rigorous investigations—a rather exceptional situation in the field of correlated quantum-particle systems. From a technical point of view, the standard spinless Falicov–Kimball model (such as that studied in Refs. 9, 10) has been modified in order to allow for segregated phases, in the regime specified above. That is, the Hamiltonian has been augmented by a direct, Ising-like, nearest-neighbor (n.n.) attractive interaction between the immobile particles. However, from the point of view of the crystallization problem, where the immobile particles play the role of ions,⁽¹⁰⁾ the added attractive interaction is a physically motivated supplement, representing Van der Waals interactions, to hardcore and (screened) Coulomb interactions, that are already incorporated in the spinless Falicov–Kimball model.

We have succeeded in proving that charge-stripe phases are stable in some domains of phase diagrams. By varying slightly the interaction parameter of the Ising-like n.n. interaction, the system is driven from a crystalline (chessboard) phase to a segregated phase, via quasi-one-dimensional striped phases.

In our previous studies⁽¹⁾ as well as in this paper, we have included also a subsidiary direct interaction between the immobile particles, an Ising-like next nearest-neighbor (n.n.n.) interaction, much weaker than the n.n. interaction. This interaction can reinforce or frustrate the n.n. interaction, depending on the sign of its interaction constant. We find it useful in discussing similarities and differences between the cases of spinless fermions and hardcore bosons.

The obtained results on the stability of striped phases open new possibilities of investigating various characteristics of these phases. Since it is the role of hopping anisotropy that has received a considerable attention in the recent literature of the subject,^(4,11–13) we have addressed this problem, and in this paper we discuss the role of a weak anisotropy of n.n. hopping.

The paper is organized as follows. In next section we define the model studied in this paper, describe symmetries of its grand-canonical phase diagrams, and introduce strong-coupling expansions of the ground-state energy (effective Hamiltonians) and the corresponding phase diagrams. Then, in Sec. 3, we carry out a detailed analysis of phase diagrams due to truncated effective Hamiltonians of orders not exceeding four, in presence of a weak anisotropy of n.n. hopping. In Sec. 4, we draw conclusions and provide a summary. Various technical details are placed in Appendices A, B, and C.

2. THE MODEL AND ITS GROUND-STATE PHASE DIAGRAM—BASIC PROPERTIES

2.1. The Model

The model to be studied is a simplified version of the one band, spin 1/2 Hubbard model, known as the static approximation (one sort of electrons hops while the other sort is immobile), augmented by a direct Ising-like interaction V between the immobile particles. Only a n.n. hopping is taken into account and we allow for its anisotropy. The total Hamiltonian of the system reads:

$$H_0 = H_{FK} + V, \quad (1)$$

$$H_{FK} = -t_h \sum_{\langle x,y \rangle_{1,h}} (c_x^+ c_y + c_y^+ c_x) - t_v \sum_{\langle x,y \rangle_{1,v}} (c_x^+ c_y + c_y^+ c_x) + U \sum_x \left(c_x^+ c_x - \frac{1}{2} \right) s_x, \quad (2)$$

$$V = \frac{W}{8} \sum_{\langle x,y \rangle_1} s_x s_y - \frac{\tilde{\varepsilon}}{16} \sum_{\langle x,y \rangle_2} s_x s_y. \quad (3)$$

In the above formulae, the underlying lattice is a square lattice, denoted Λ , consisting of sites x, y, \dots , whose number is $|\Lambda|$, having the shape of a $\sqrt{|\Lambda|} \times \sqrt{|\Lambda|}$ torus. In (2,3), and in other formulae below, the sums $\sum_{\langle x,y \rangle_i}$, $\sum_{\langle x,y \rangle_{i,h}}$, or $\sum_{\langle x,y \rangle_{i,v}}$, with i natural, stand for the summation over all the i -th order n.n. pairs of lattice sites in Λ , directed horizontally (h) or vertically (v), respectively, with each pair counted once.

The subsystem of mobile spinless particles (here-after called *the electrons*) is described in terms of creation and annihilation operators of an electron at site x , c_x^+ , c_x , respectively, satisfying the canonical anticommutation relations (spinless fermions) or the commutation relations of spin 1/2 operators S_x^+ , S_x^- , S_x^z (hardcore bosons). The total electron-number operator is $N_e = \sum_x c_x^+ c_x$, and (with a little abuse of notation) the corresponding electron density is $\rho_e = N_e/|\Lambda|$. There is no direct interaction between the mobile particles. Their energy is due to the anisotropic hopping, with t_h (t_v) being the n.n. horizontal (vertical) hopping intensity, and due to the interaction with the localized particles, whose strength is controlled by the coupling constant U .

The Hamiltonian H_{FK} is well-known as the Hamiltonian of the spinless Falicov–Kimball model, a simplified version of the Hamiltonian put forward in Ref. 14. A review of rigorous results concerning this model and an extensive list of relevant references can be found in Refs. 15–17.

The subsystem of localized particles (here-after called *the ions*), is described by a collection of pseudo-spins $\{s_x\}_{x \in \Lambda}$, with $s_x = 1, -1$ ($s_x = 1$ if the site x is occupied by an ion and $s_x = -1$ if it is empty), called the *ion configurations*. The total number of ions is $N_i = \sum_x (s_x + 1)/2$ and the ion density is $\rho_i = N_i/|\Lambda|$. In our model the ions interact directly by means of the isotropic Ising-like interaction V .

Clearly, in the composite system, whose Hamiltonian is given by (1) with an arbitrary electron-ion coupling U , the particle-number operators N_e, N_i , and pseudo-spins s_x , are conserved. Therefore, the description of the classical subsystem in terms of the ion configurations $S = \{s_x\}_{x \in \Lambda}$ remains valid. Whenever periodic configurations of pseudo-spins are considered, it is assumed that Λ is sufficiently large, so that it accommodates an integer number of elementary cells.

2.2. The Ground-State Phase Diagram in the Grand-Canonical Ensemble

In what follows, we shall study the ground-state phase diagram of the system defined by (1) in the grand-canonical ensemble. That is, let

$$H(\mu_e, \mu_i) = H_0 - \mu_e N_e - \mu_i N_i, \quad (4)$$

where μ_e, μ_i are the chemical potentials of the electrons and ions, respectively, and let $E_S(\mu_e, \mu_i)$ be the ground-state energy of $H(\mu_e, \mu_i)$, for a given configuration S of the ions. Then, the ground-state energy of $H(\mu_e, \mu_i)$, $E_G(\mu_e, \mu_i)$, is defined as $E_G(\mu_e, \mu_i) = \min\{E_S(\mu_e, \mu_i) : S\}$. The minimum is attained at the set G of the ground-state configurations of ions. We shall determine the subsets of the space of the Hamiltonian's energy parameters, where G consists of periodic configurations of ions, uniformly in the size of the underlying square lattice.

In studies of grand-canonical phase diagrams an important role is played by unitary transformations (*hole-particle transformations*) that exchange particles and holes, $c_x^+ c_x \rightarrow 1 - c_x^+ c_x$, $s_x \rightarrow -s_x$, and for some (μ_e^0, μ_i^0) leave the Hamiltonian $H(\mu_e, \mu_i)$ invariant. For the mobile particles such a role is played by the transformations: $c_x^+ \rightarrow \epsilon_x c_x$, where $\epsilon_x = 1$ for bosons while for fermions $\epsilon_x = 1$ at the even sublattice of Λ and $\epsilon_x = -1$ at the odd one. Clearly, since H_0 is invariant under the joint hole-particle transformation of mobile and localized particles, $H(\mu_e, \mu_i)$ is hole-particle invariant at the point $(0, 0)$. At the hole-particle symmetry point, the system under consideration has very special properties, which simplify studies of its phase diagram.⁽¹⁰⁾ Moreover, by means of the defined above hole-particle transformations one can determine a number of symmetries of the grand-canonical phase diagram.⁽¹⁸⁾ The peculiarity of the model is that the case of attraction ($U < 0$) and the case of repulsion ($U > 0$) are related by a unitary

transformation (the hole-particle transformation for ions): if $S = \{s_x\}_{x \in \Lambda}$ is a ground-state configuration at (μ_e, μ_i) for $U > 0$, then $-S = \{-s_x\}_{x \in \Lambda}$ is the ground-state configuration at $(\mu_e, -\mu_i)$ for $U < 0$. Consequently, without any loss of generality one can fix the sign of the coupling constant U . Moreover (with the sign of U fixed), there is an *inversion symmetry* of the grand-canonical phase diagram, that is, if S is a ground-state configuration at (μ_e, μ_i) , then $-S$ is the ground-state configuration at $(-\mu_e, -\mu_i)$. Therefore, it is enough to determine the phase diagram in a half-plane specified by fixing the sign of one of the chemical potentials.

Our aim in this paper is to investigate the ground-state phase diagrams of our systems, for general values of the energy parameters that appear in $H(\mu_e, \mu_i)$. According to the state of art, this is feasible only in the *strong-coupling regime*, i.e. when $|t/U|$ is sufficiently small. Therefore, from now on we shall consider exclusively the case of a large positive coupling U , and we express all the parameters of $H(\mu_e, \mu_i)$ in the units of U , preserving the previous notation.

2.3. The Ground-State Energy and Phase Diagram in the Strong-Coupling Regime

In the strong-coupling regime, the ground-state energy $E_S(\mu_e, \mu_i)$ can be expanded in a power series in t . One of the ways to achieve this, for fermions and for hardcore bosons, is a method of unitarily equivalent interactions.⁽¹⁹⁾ The result, with the expansion terms up to order four shown explicitly, reads:

$$\begin{aligned}
 E_S^f(\mu_e, \mu_i) &= (E_S^f)^{(4)}(\mu_e, \mu_i) + (R_S^f)^{(4)}, \\
 (E_S^f)^{(4)}(\mu_e, \mu_i) &= -\frac{1}{2}(\mu_i - \mu_e) \sum_x (s_x + 1) \\
 &\quad + \left[\frac{t^2}{4} - \frac{3t^4}{16} - \frac{3}{8}\gamma t^4 + \frac{W}{8} \right] \sum_{\langle x,y \rangle_{1,h}} s_x s_y \\
 &\quad + \left[\gamma \frac{t^2}{4} - \frac{3}{8}\gamma t^4 - \gamma^2 \frac{3t^4}{16} + \frac{W}{8} \right] \sum_{\langle x,y \rangle_{1,v}} s_x s_y \\
 &\quad + \left[\gamma \frac{3t^4}{16} - \frac{\tilde{\varepsilon}}{16} \right] \sum_{\langle x,y \rangle_2} s_x s_y + \frac{t^4}{8} \sum_{\langle x,y \rangle_{3,h}} s_x s_y \\
 &\quad + \gamma^2 \frac{t^4}{8} \sum_{\langle x,y \rangle_{3,v}} s_x s_y + \gamma \frac{t^4}{16} \sum_P (1 + 5s_P), \tag{5}
 \end{aligned}$$

in the case of hopping fermions, and

$$\begin{aligned}
 E_S^b(\mu_e, \mu_i) &= (E_S^b)^{(4)}(\mu_e, \mu_i) + (R_S^b)^{(4)}, \\
 (E_S^b)^{(4)}(\mu_e, \mu_i) &= -\frac{1}{2}(\mu_i - \mu_e) \sum_x (s_x + 1) \\
 &\quad + \left[\frac{t^2}{4} - \frac{3t^4}{16} - \frac{1}{8}\gamma t^4 + \frac{W}{8} \right] \sum_{\langle x,y \rangle_{1,h}} s_x s_y \\
 &\quad + \left[\gamma \frac{t^2}{4} - \frac{1}{8}\gamma t^4 - \gamma^2 \frac{3t^4}{16} + \frac{W}{8} \right] \sum_{\langle x,y \rangle_{1,v}} s_x s_y \\
 &\quad + \left[\gamma \frac{5t^4}{16} - \frac{\tilde{\varepsilon}}{16} \right] \sum_{\langle x,y \rangle_2} s_x s_y + \frac{t^4}{8} \sum_{\langle x,y \rangle_{3,h}} s_x s_y \\
 &\quad + \gamma^2 \frac{t^4}{8} \sum_{\langle x,y \rangle_{3,v}} s_x s_y - \gamma \frac{t^4}{16} \sum_P (5 + s_P), \tag{6}
 \end{aligned}$$

in the case of hopping hardcore bosons. The above expressions are given up to a term independent of the ion configurations and the chemical potentials. In (5) and (6), we have set $t_h \equiv t$, and $t_v = \sqrt{\gamma}t$, with $0 \leq \gamma \leq 1$, P denotes a (2×2) -sites plaquette of the square lattice Λ , s_P stands for the product of pseudo-spins assigned to the corners of P . The remainders, $(R_S^f)^{(4)}$ and $(R_S^b)^{(4)}$, which are independent of the chemical potentials and the parameters W and $\tilde{\varepsilon}$, but dependent on γ , collect those terms of the expansions that are proportional to t^{2m} , with $m = 3, 4, \dots$. It can be proved that the above expansions are absolutely convergent, uniformly in Λ , provided that $t < 1/16$ and $|\mu_e| < 1 - 16t$.⁽²⁰⁾ Moreover, under these conditions the ground-state densities of particles satisfy the half-filling relation: $\rho_e + \rho_i = 1$.

We note that on taking into account the inversion symmetry of the phase diagram in the (μ_e, μ_i) -plane and the fact that the ground-state energies depend only on the difference of the chemical potentials, in order to determine the phase diagram in the stripe $|\mu_e| < 1 - 16t$ it is enough to consider the phase diagram at the half-line $\mu_e = 0$, $\mu_i < 0$ (or $\mu_i > 0$). At this half-line we set $\mu_i \equiv \mu$.

3. PHASE DIAGRAMS ACCORDING TO TRUNCATED EXPANSIONS

Due to the convergence of the expansions (5) and (6), it is possible to establish rigorously a part of the phase diagram, that is the ground-state configurations of ions are determined in the space of the Hamiltonian's energy parameters everywhere, except some small regions. This is accomplished by determining the ground-state phase diagram of the expansion truncated at the order k (*the k -th order phase diagram*), that is according to the k -th order effective Hamiltonians

$(E_S^f)^{(k)}(0, \mu)$ and $(E_S^b)^{(k)}(0, \mu)$. The remainder of the expansion, that consists of a term of the next order, $k' > k$, and other higher-order terms, cannot modify the obtained phase diagram, except some narrow regions along the phase boundaries whose width is of the order k' . The procedure of constructing the phase diagram is recursive: the phase diagram of the effective interaction truncated at the order k' is constructed on the basis of the phase diagram obtained at the preceding order k . That is, the conditions imposed on the ground-state configurations by the k -th order effective Hamiltonian have to be obeyed by the ground-state configurations of the k' -th order effective Hamiltonian, since the k' -th order terms of the expansion cannot change the hierarchy of the configuration's energies established by the k -th order effective Hamiltonian (they can only split the energies of configurations in cases of degeneracy). At each step of the construction, we use the m -potential method introduced in Ref. 21, with technical developments described in Refs. 18, 20, 22. For details, like the so called zero potentials, specific to this paper, see Ref. 1 and Appendix A.

In Ref. 1 we have obtained the ground-state phase diagrams, according to the fourth-order isotropic effective Hamiltonians $(E_S^f)^{(4)}(0, \mu)|_{\gamma=1}$ and $(E_S^b)^{(4)}(0, \mu)|_{\gamma=1}$, for hole-particle symmetric systems ($\mu = 0$) with a weak (fourth order) n.n.n. subsidiary interaction, and for unsymmetrical systems without the subsidiary interaction ($\tilde{\varepsilon} = 0$), see the top phase diagrams in Figs. 1, 2 and 4, where the parameters β , ϵ , and ω are given by Eqs. (7) and (9).

Second-order phase diagrams of Ref. 1 consist exclusively of phases whose configurations are invariant with respect to $\pi/2$ -rotations, with *macroscopic degeneracies* (i.e. the number of configurations grows exponentially with the number of sites) at the boundaries of phase domains. Stripe phases, whose configurations are not invariant with respect to $\pi/2$ -rotations, appear on perturbing the second-order phase diagrams by the fourth-order isotropic interactions. Here, we would like to observe the influence of a weak anisotropy of n.n. hopping on these stripe phases. Since we are working with truncated effective Hamiltonians, we have to assign an order to the deviation of the anisotropy parameter γ from the value 1 (corresponding to the isotropic case). Therefore, we introduce the anisotropy order, a , and the new anisotropy parameter, β_a :

$$\gamma = 1 - \beta_a t^a. \quad (7)$$

The orders of the deviations can be neither too small, not to modify the second-order effective Hamiltonians, nor too high, to effect the considered effective Hamiltonian of the highest order. Since here, the highest order of the effective Hamiltonians is $k = 4$, the weakest admissible deviation from the isotropic case corresponds to the highest anisotropy order $a = 2$. Then, we can consider an intermediate deviation, i.e. $0 < a < 2$. The strongest deviation, i.e. $a = 0$, is not admissible, since it modifies the second-order effective Hamiltonians.

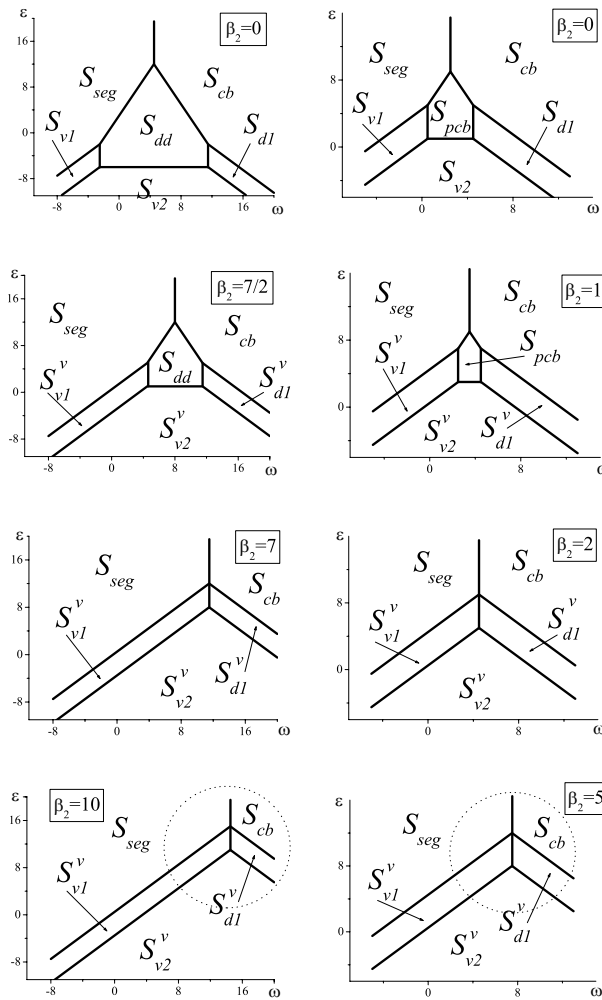


Fig. 1. The case of the weakest anisotropy and hole-particle symmetric systems ($\mu = 0$). The parameters labeling the axes are given by (9). Phase diagrams of $(H_T^f)^{(4)}$ (given by (11))—left column, and $(H_T^b)^{(4)}$ —right column, for an increasing sequence of values of β_2 . The representative ion configurations of the displayed phases are shown in Fig. 5 (for more comments see Sec. 3). For fermions, the critical value is $\beta_2 = 7$, while for bosons it is $\beta_2 = 2$. The equations defining the boundary lines of the phase domains are given in Tables II and III of Appendix B, while the corresponding zero potentials—in Appendix A and in Tables IX–XVI of Appendix C. In the bottom diagrams, the regions surrounded by dotted circles, are reconsidered in the case of an intermediate anisotropy.

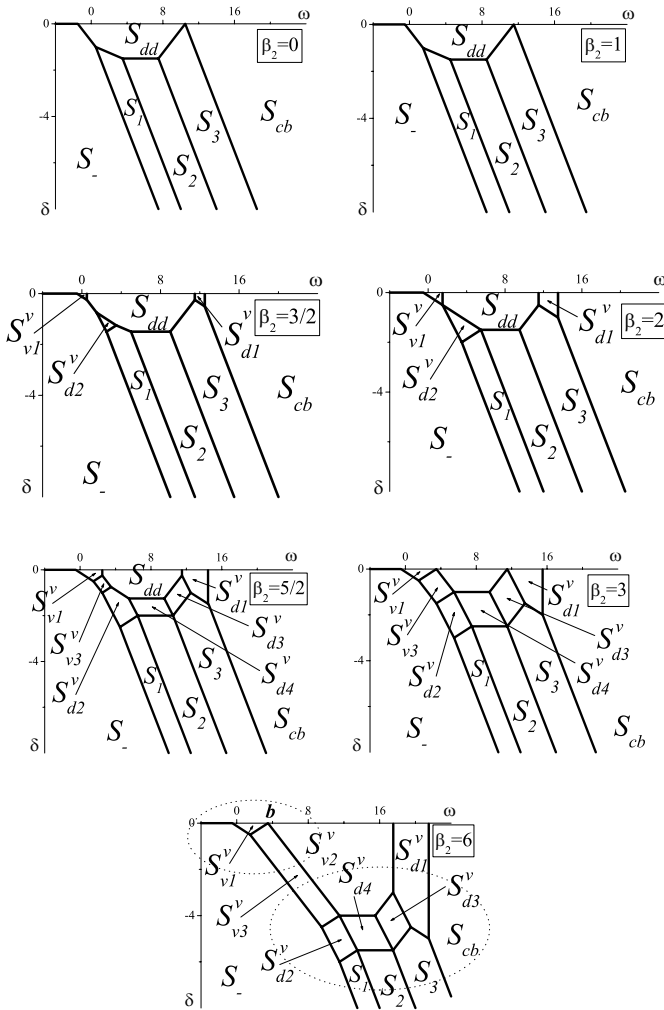


Fig. 2. The case of the weakest anisotropy, off the hole-particle symmetry, with $\tilde{\epsilon} = 0$. The parameters labeling the axes are given by (9). The phase diagram of $(H_T^f)^{(4)}$ (given by (11)) for an increasing sequence of values of β_2 . The representative ion configurations of the displayed phases are shown in Fig. 5 (for more comments see Sec. 3). The critical values are: $\beta_2 = 1$, $\beta_2 = 2$, and $\beta_2 = 3$. In the blank region of the phase diagram, for the critical $\beta_2 = 3$, there are numerous ground-state configurations. The set of these configurations comprises the configurations of S_{v2}^v and S_{dd} , and many other configurations. All of them can be built of the following T -plaquette configurations: 26, 52, 53, 68, 81, 88, 101, 116, 117, 143 (see Fig. 3). The equations defining the boundary lines of the phase domains are given in Table IV of Appendix B, while the corresponding zero potentials—in Appendix A and in Tables XVII–XXIII of Appendix C. In the bottom diagram, the regions surrounded by dotted ellipses are reconsidered in the case of an intermediate anisotropy.

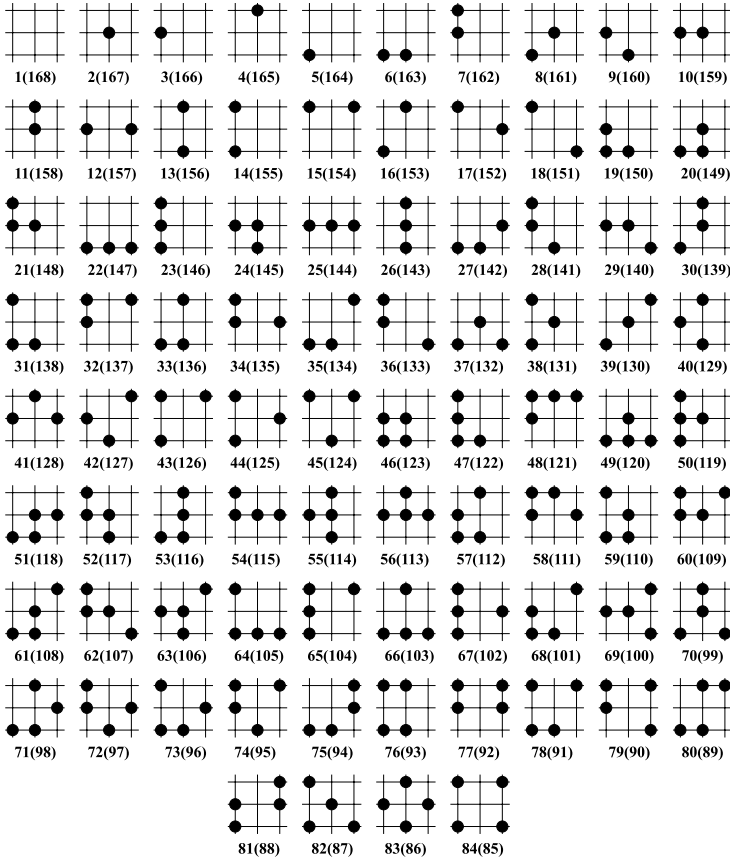


Fig. 3. All the T -plaquette configurations in the case of anisotropic hopping, up to rotations by π and reflections in lattice lines parallel to the axes. The configurations that can be obtained from the displayed ones by the hole-particle transformation are not shown, only the numbers assigned to them are given in the brackets.

3.1. The Smallest Deviation from the Isotropic Case

In the sequel, we drop the arguments of ground-state energies, that is we set $(E_S^f)^{(4)}(0, \mu) \equiv (E_S^f)^{(4)}$, etc. In the case of the smallest deviation from the isotropic case, the fourth-order effective Hamiltonian for fermions reads:

$$(E_S^f)^{(4)} = (E_S^f)^{(4)} \Big|_{\gamma=1} - \beta_2 \frac{t^4}{4} \sum_{\langle x,y \rangle_{1,\nu}} s_x s_y, \tag{8}$$

while for hardcore bosons, only the first term, representing the isotropic fourth-order effective Hamiltonian, has to be changed properly.

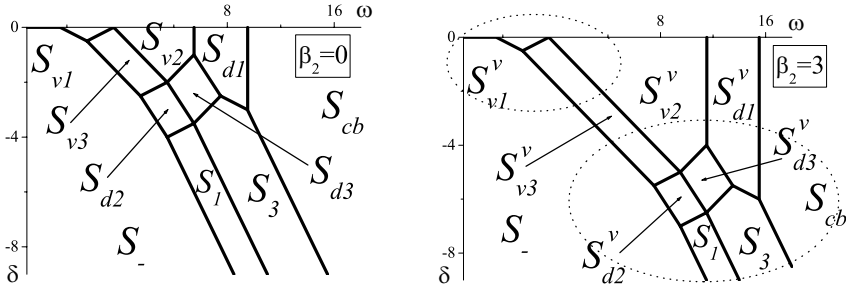


Fig. 4. The case of the weakest anisotropy, off the hole-particle symmetry, with $\varepsilon = 0$. The parameters labeling the axes are given by (9). The phase diagram of $(H_T^f)^{(4)}$ (given by (11)); the isotropic diagram ($\beta_2 = 0$) and the anisotropic diagram ($\beta_2 = 3$). The representative ion configurations of the displayed phases are shown in Fig. 5 (for more comments see Sec. 3). Here, no critical values have been detected. The equations defining the boundary lines of the phase domains are given in Table V of Appendix B, while the corresponding zero potentials—in Appendix A and in Tables XXIV and XXV of Appendix C. In the right diagram, the regions surrounded by dotted ellipses are reconsidered in the case of an intermediate anisotropy.

Clearly, the effective Hamiltonians at order zero and two are isotropic, and there is no difference between the cases of hopping fermions and hopping bosons. The corresponding phase diagrams⁽¹⁾ are built of three *phases* S_+ , S_- , and S_{cb} , i.e. of the sets of configurations with the same minimal energy in some open regions of phase diagrams (*phase domains*). By the same symbols as the phases we denote also their domains. The phase S_+ (S_-) consists of a single translation invariant configuration, which is completely filled with the ions (completely empty). A representative configuration of the chessboard phase, S_{cb} , is shown in Fig. 5. The three named phases coexist at the open half-line $W = -2t^2$, $\mu = 0$, $\tilde{\varepsilon} > 0$. However, if $\tilde{\varepsilon}$ attains zero, the finite (independent of the size of Λ) degeneracy of the open half-line changes into a macroscopic one. It is the point $W = -2t^2$, $\mu = 0$, $\tilde{\varepsilon} = 0$ around which the fourth-order diagrams, containing various stripe phases, have been constructed in Ref. 1. Now, to observe the effect of the fourth-order anisotropy term of (8) the construction of the fourth-order diagrams has to be carried out again. As in the isotropic case, this is facilitated by introducing new variables, ω , δ , and ε :

$$W = -2t^2 + t^4\omega, \quad \mu = t^4\delta, \quad \tilde{\varepsilon} = t^4\varepsilon, \quad (9)$$

and rewriting the fourth-order effective Hamiltonian in the form,

$$(E_S^f)^{(4)} = \frac{t^4}{2} \sum_T (H_T^f)^{(4)}, \quad (10)$$

where

$$(H_T^f)^{(4)} = (H_T^f)^{(4)} \Big|_{\gamma=1} - \frac{\beta_2}{12} \sum_{(x,y)_{1,v}}'' s_x s_y, \quad (11)$$

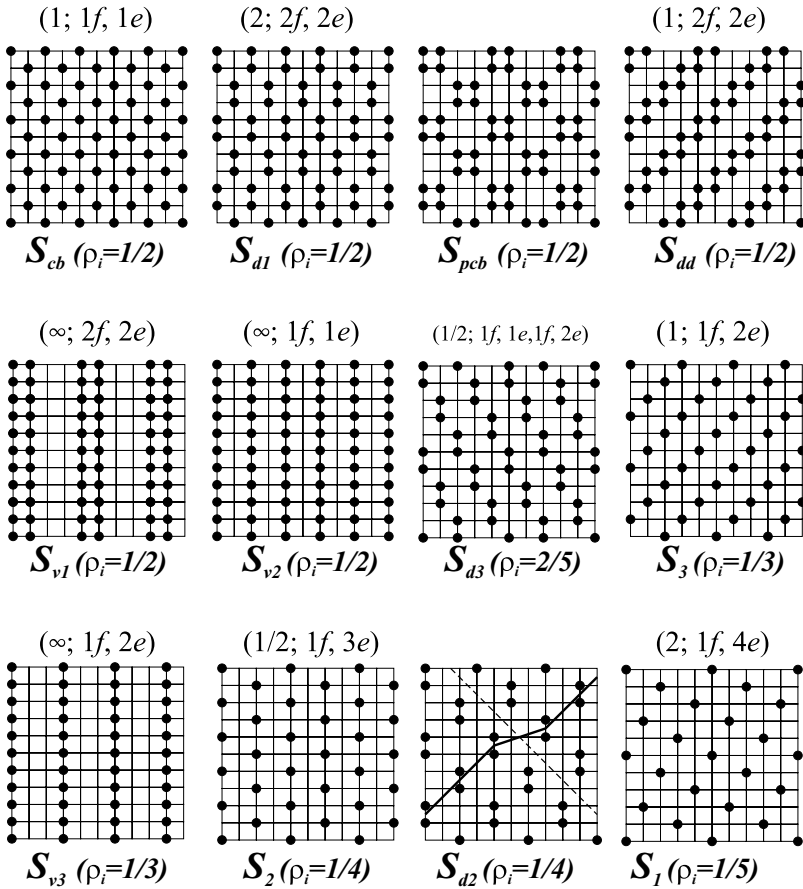


Fig. 5. The representative configurations of the phases that appear in the phase diagrams of $H_0|_{\gamma=1}$. The remaining configurations can be obtained by applying the spatial symmetries of $.H_0|_{\gamma=1}$. As a representative configuration of the set S_{d2} , whose degeneracy grows like $\exp(const\sqrt{N})$, we show a configuration with one defect line (the dashed line); the continuous line is a guide for the eye. Explanations of the symbols on the top of the representative configurations and other comments are given in Sec. 3.

with analogous expressions in the bosonic case, and with the isotropic potentials $(H_T^f)^{(4)}|_{\gamma=1}, (H_T^b)^{(4)}|_{\gamma=1}$, given by

$$\begin{aligned}
 (H_T^f)^{(4)}|_{\gamma=1} &= -\delta(s_5 + 1) + \frac{1}{24} \left(\omega - \frac{9}{2} \right) \sum_{\langle x,y \rangle_1}'' s_x s_y + \frac{1}{32} (3 - \varepsilon) \sum_{\langle x,y \rangle_2}'' s_x s_y \\
 &+ \frac{1}{12} \sum_{\langle x,y \rangle_3}'' s_x s_y + \frac{1}{32} \sum_P'' (5s_P + 1), \tag{12}
 \end{aligned}$$

$$\begin{aligned}
 (H_T^b)^{(4)} \Big|_{\gamma=1} = & -\delta (s_5 + 1) + \frac{1}{24} \left(\omega - \frac{5}{2} \right) \sum''_{\langle x,y \rangle_1} s_x s_y + \frac{1}{32} (5 - \varepsilon) \sum''_{\langle x,y \rangle_2} s_x s_y \\
 & + \frac{1}{12} \sum''_{\langle x,y \rangle_3} s_x s_y - \frac{1}{32} \sum''_P (5 + s_P). \tag{13}
 \end{aligned}$$

In the above formulae T stands for a (3×3) -sites plaquette of a square lattice, later on called *the T -plaquette*, and s_5 stands for the central site of a T -plaquette. The summation in double-primed sums runs over a T -plaquette.

We would like to get an idea of the phase diagram in the space of the four energy parameters $(\omega, \varepsilon, \beta_2, \delta)$, which appear in the Hamiltonian. Due to the fact that the domains occupied by the phases are polyhedral sets (see Appendix A), this goal can be achieved by studying phase diagrams in some two-dimensional hyperplanes. Of particular interest are those hyperplanes that result from intersecting the four-dimensional space by the hyperplanes $\delta = 0$ and $\beta_2 = \text{const}$, and by the hyperplanes $\varepsilon = 0$ and $\beta_2 = \text{const}$. A collection of such sections, for suitable values of β_2 forming a finite increasing from zero sequence, enables us to observe how the anisotropy effects our system if it is hole-particle symmetric (Fig. 1) and if it is not (Fig. 2—for fermions, Fig. 4—for bosons.).

The phases that appear in these phase diagrams can conveniently be described in terms of the phases found in ground-state phase diagrams of the isotropic systems with Hamiltonian $H_0|_{\gamma=1}^{(1)}$ (see Fig. 5). The phases of anisotropic systems, considered here, either coincide with or are simple modifications of the isotropic phases. In fact, in Fig. 5 we display only representative configurations of the phases of isotropic systems. The remaining configurations can be obtained by means of the spatial symmetry operations of $H_0|_{\gamma=1}$ (translations and rotations by $\pi/2$). In Table I, for each phase that appears in the displayed phase diagrams, we give the labels of those T -plaquette configurations (Fig. 3) that are obtained by restricting the configurations of this phase to a T -plaquette. In the process of constructing phase diagrams according to the fourth-order effective Hamiltonians, one obtains first these T -plaquette configurations, and then, out of them, one reconstructs the configurations of a phase (see Refs. 1, 20). Moreover, knowledge of the T -plaquette configurations of two neighboring phases is necessary for controlling the degeneracy of ground-state configurations at the boundary between these phases.

Among the phases of isotropic systems, we can distinguish the class of the *dimeric phases*, $\mathcal{S}_{d1}, \dots, \mathcal{S}_{d4}$, and the class of the *axial-stripe phases*, $\mathcal{S}_{v1}, \mathcal{S}_{v2}$, and \mathcal{S}_{v3} . The configurations of the dimeric phases consist of isolated pairs of n.n. occupied sites (*dimers*). In the configurations of the axial-stripe phases, the ions fill completely some, parallel to one of the axes, lattice lines, so that a periodic pattern of *stripes* is formed. Out of the dimeric or axial-stripe configurations, shown in Fig. 5, only those with dimers or stripes oriented vertically appear in the

Table I. Labels of T -plaquette configurations (Fig. 3) that are obtained by restricting the configurations of a phase to a T -plaquette.

Phase	Labels of T -plaquette configurations
\mathcal{S}_{seg}	1, 168
\mathcal{S}_{cb}	83, 86
\mathcal{S}_{dd}	68, 101
\mathcal{S}_{pcb}	75, 94
\mathcal{S}_-	1
\mathcal{S}_1	2, 16, 17
\mathcal{S}_2	2, 12, 13, 44, 45
\mathcal{S}_3	39, 42
\mathcal{S}_{v1}	22, 23, 146, 147
\mathcal{S}_{v1}^v	23, 146
\mathcal{S}_{v2}	25, 26, 143, 144
\mathcal{S}_{v2}^v	26, 143
\mathcal{S}_{v3}	22, 23, 25, 26
\mathcal{S}_{v3}^v	23, 26
\mathcal{S}_{d1}	69, 70, 99, 100
\mathcal{S}_{d1}^v	70, 99
\mathcal{S}_{d2}	3, 4, 6, 7, 10, 11, 16, 17, 35, 36
\mathcal{S}_{d2}^v	4, 7, 11, 16, 36
\mathcal{S}_{d3}	29, 30, 73, 74, 80, 81
\mathcal{S}_{d3}^v	30, 74, 81
\mathcal{S}_{d4}^v	11, 13, 45, 77, 79, 81

phase diagrams of anisotropic systems. Such a restricted phases are marked in our anisotropic phase diagrams by the additional superscript, v .

All the phases presented in Fig. 5, except \mathcal{S}_{d2} , contain exclusively periodic configurations, related by the spatial symmetries of $H_0|_{\gamma=1}$, hence of the degeneracy independent of the size of the underlying lattice. The set \mathcal{S}_{pcb} (of plaquette-chessboard configurations) contains configurations built of elementary plaquettes with occupied sites, forming a square lattice according to the same rules as filled sites form a square lattice in the chessboard configurations of \mathcal{S}_{cb} . The remaining phases consist of configurations that have a *quasi-one-dimensional* structure. That is, they are built of sequences of completely filled with ions lattice lines of given slope. Such a configuration can be specified by giving the slope of the filled lattice lines of a representative configuration and than the succession of filled (f) and empty (e) consecutive lattice lines in a period. For instance, the representative configuration of \mathcal{S}_{d1} (see Fig. 5) is built of filled lines with the slope 2 and, in the period, two consecutive filled lines are followed by two empty lines, which is denoted (2; 2f, 2e). This kind of description of the remaining quasi-one-dimensional

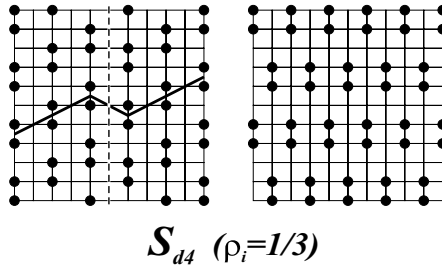


Fig. 6. The representative configurations of the set \mathcal{S}_{d4}^v whose degeneracy grows like $\exp(const\sqrt{\Lambda})$. The left configuration is an example of configurations with defect lines. Here, there is one defect line (the dashed line, the continuous line is a guide for the eye): the vertical lattice line separating a periodic configuration of vertical dimers from its vertical translate by one lattice constant. The right configuration is a periodic configuration of dimers. For more comments see Sec. 3.

configurations is given in Fig. 5. The phase \mathcal{S}_{dd} , shown in Fig. 5, is an example of a *diagonal-stripe phase*.

The structure of the phases \mathcal{S}_{d2} , \mathcal{S}_{d2}^v , \mathcal{S}_{d4} , and \mathcal{S}_{d4}^v is a lot more complex, since their degeneracy grows indefinitely with the size of the lattice. They appear exclusively in the phase diagrams off the hole-particle symmetry. Specifically, \mathcal{S}_{d2}^v is stable in the anisotropic phase diagrams of hopping fermions, Fig. 2, and in the anisotropic phase diagrams of hopping bosons, Fig. 4. The phase \mathcal{S}_{d2} is stable only in the isotropic phase diagram of hopping bosons, while \mathcal{S}_{d4}^v , whose representative configurations are displayed in Fig. 6,—only in the anisotropic phase diagrams of hopping fermions with sufficiently large hopping anisotropy, Fig. 2.

In \mathcal{S}_{d2} one can distinguish two classes, \mathcal{S}_{d2a} and \mathcal{S}_{d2b} , of periodic configurations with parallelogram elementary cells. A configuration in \mathcal{S}_{d2a} consists of vertical (horizontal) dimers of occupied sites that form a square lattice, where the sides of the elementary squares have the length $2\sqrt{2}$ and the slope ± 1 . In a configuration of \mathcal{S}_{d2b} , the elementary parallelograms formed by dimers have the sides of the length $2\sqrt{2}$ and the slope ± 1 , and the sides of the length $\sqrt{10}$ and the slope $\pm 1/3$. Two configurations, one from \mathcal{S}_{d2a} and one from \mathcal{S}_{d2b} , having the same kind of dimers (vertical or horizontal), can be merged together along a “defect line” of the slope ± 1 (the dashed line), as shown in Fig. 5, without increasing the energy. By introducing more defect lines one can construct many ground-state configurations whose number scales with the size of the lattice as $\exp(const\sqrt{\Lambda})$. The same kind of degeneracy is in \mathcal{S}_{d4} . This phase consists of three classes of periodic configurations of dimers.

In the class \mathcal{S}_{d4a} , the elementary cell can be chosen as a parallelogram whose two sides of length 3 are parallel to dimers (which are vertical or horizontal). If the dimers are oriented vertically, then the other two sides have the slope $1/2$ and the length $\sqrt{5}$. By reflecting an elementary cell of \mathcal{S}_{d4a} in a lattice line passing through

its side that is parallel to dimers, we obtain an elementary cell of the class \mathcal{S}_{d4b} . In the third class, \mathcal{S}_{d4c} , an elementary cell can be chosen as a rhomb formed by the centers of dimers, with the sides of length $\sqrt{10}$. Two configurations, one from \mathcal{S}_{d4a} and one from \mathcal{S}_{d4b} , having the same kind of dimers (vertical or horizontal), can be merged together along a “defect line” parallel to dimers (dashed line in Fig. 6) without increasing the energy. In this way, a numerous family of configurations can be constructed, with the number of configurations growing like $\exp(\text{const}\sqrt{\Lambda})$.

It follows from the polyhedral shape of the phase domains, that the set of values of β_2 is partitioned into open intervals, given by *the critical values* of the anisotropy. On varying β_2 inside such an interval, the boundary lines of phase domains undergo only translations. As a critical value of anisotropy is approached, some boundary lines merge into a line or a point, which results in the disappearance of some phase domains. And vice versa, some points and lines break off, creating new phase domains.

In particular, in the hole-particle symmetric case it can be inferred from Fig. 1 that at least up to $\beta_2 = 10$ —for hopping fermions, and at least up to $\beta_2 = 5$ —for hardcore bosons, there is only one critical value of β_2 . For fermions, it amounts to $\beta_2 = 7$, where the phase \mathcal{S}_{dd} disappears, while for hardcore bosons it is $\beta_2 = 2$, where the phase \mathcal{S}_{pcb} disappears.

Off the hole-particle symmetry, there are no critical values of β_2 for bosons, at least up to $\beta_2 = 3$. For fermions, if $\beta_2 \leq 6$, there are as many as three critical values of β_2 : $\beta_2 = 1$, where the phases \mathcal{S}_{v1}^v , \mathcal{S}_{d1}^v , and \mathcal{S}_{d2}^v appear, $\beta_2 = 2$, where the phases \mathcal{S}_{d3}^v , \mathcal{S}_{d4}^v , and \mathcal{S}_{v3}^v appear, and $\beta_2 = 3$, where \mathcal{S}_{dd} is replaced by \mathcal{S}_{v2}^v .

To verify whether for the values of β_2 larger than those considered in this section some new critical values appear, we proceed to investigating stronger, i.e. intermediate, deviations from the isotropic case.

3.2. The Intermediate Deviation from the Isotropic Case

Now, the fourth-order effective Hamiltonian for fermions assumes the form:

$$(E_S^f)^{(4)} = (E_S^f)^{(4)}\Big|_{\gamma=1} - \beta_2 \frac{t^{2+a}}{4} \sum_{\langle x,y \rangle_{1,v}} s_x s_y, \quad (14)$$

and for hardcore bosons an analogous formula holds true. Obviously, the phase diagrams in the zeroth and second orders remain the same as in the isotropic case, described above. Therefore, we proceed to constructing the phase diagram in next order, which is $(2+a)$ -order with $0 < a < 2$, and the corresponding effective Hamiltonian reads:

$$E_S^{(2+a)} = E_S^{(2)} - \beta_2 \frac{t^{2+a}}{4} \sum_{\langle x,y \rangle_{1,v}} s_x s_y, \quad (15)$$

where $E_S^{(2)}$ stands for the, common for fermions and bosons, second-order effective Hamiltonian. For the reasons given in the previous subsection, we consider a neighborhood of the point $W = -2t^2$, $\mu = 0$, $\tilde{\varepsilon} = 0$, where the energies of all the configurations are equal.⁽¹⁾ In this neighborhood it is convenient to introduce new variables, δ' , ε' , and ω' ,

$$\mu = t^{2+a} \delta', \quad \tilde{\varepsilon} = t^{2+a} \varepsilon', \quad W = -2t^2 + t^{2+a} \omega', \quad (16)$$

and rewrite the expansion up to the order $2 + a$:

$$E_S^{(2+a)} = \frac{t^{2+a}}{2} \left\{ -\delta' \sum_x (s_x + 1) + \frac{\omega'}{4} \sum_{\langle x,y \rangle_1} s_x s_y - \frac{\varepsilon'}{8} \sum_{\langle x,y \rangle_2} s_x s_y - \frac{\beta_a}{2} \sum_{\langle x,y \rangle_{1,v}} s_x s_y \right\} = \frac{t^{2+a}}{2} \sum_P H_P^{(2+a)}, \quad (17)$$

where

$$H_P^{(2+a)} = -\frac{\delta'}{4} \sum'_x (s_x + 1) + \frac{\omega'}{8} \sum'_{\langle x,y \rangle_1} s_x s_y - \frac{\varepsilon'}{8} \sum'_{\langle x,y \rangle_2} s_x s_y - \frac{\beta_a}{4} \sum'_{\langle x,y \rangle_{1,v}} s_x s_y. \quad (18)$$

The summations in the primed sums run over a plaquette P . The plaquette potentials $H_P^{(2+a)}$ have to be minimized over all the plaquette configurations. As in the previous case, we are interested in phase diagrams of hole-particle symmetric systems ($\delta' = 0$) or unsymmetrical systems with $\varepsilon' = 0$. It appears that for such energy parameters and plaquette configurations the potentials $H_P^{(2+a)}$ are m-potentials. The resulting phase diagrams are shown in Figs. 7 and 8. We note that each of the points **a**, **b**, **c**⁻, and **c**⁺, is the coexistence point of three periodic phases. That is, the only plaquette configurations minimizing the potential $H_P^{(2+a)}$ at such a point are those obtained by restricting the configurations of coexisting phases to a plaquette.

Now, following our recursive procedure of constructing phase diagrams to some order, we are ready to investigate the effect of fourth-order interactions. As in earlier steps, it is enough to consider neighborhoods of the coexistence points **a**, **b**, and **c**⁻ of Figs. 7 and 8 (the diagram in a neighborhood of **c**⁺ can be obtained from that around **c**⁻ by a symmetry operation).

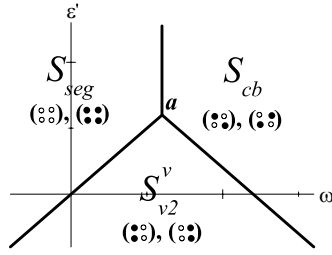


Fig. 7. The case of an intermediate anisotropy ($0 < a < 2$) and of hole-particle symmetric systems ($\mu = 0$). The phase diagram (common for hopping fermion and hardcore boson systems) of $H_P^{(2+a)}$. The coordinates of point **a** are $\omega' = \beta_a$, $\epsilon' = \beta_a$. The boundary lines of S_{v2}^v , from left to right, are: $\epsilon' = \omega'$ and $\epsilon' = -\omega' + 2\beta_a$. The boundary between S_{seg} and S_{cb} is $\omega' = \beta_a$.

A. A neighborhood of a

Here $\delta' = 0$, hence the systems are hole-particle invariant. A convenient change of variables is:

$$\omega' = \beta_a + t^{2-a}\omega, \quad \epsilon' = \beta_a + t^{2-a}\epsilon. \tag{19}$$

In these variables, the fourth-order effective Hamiltonian reads:

$$(E_S^f)^{(4)} = \frac{t^{2+a}}{2} \sum_P H_P^{(2+a)} \Big|_{\substack{\delta'=0 \\ \epsilon'=\omega'=\beta_a}} + \frac{t^4}{2} \sum_T (H_T^f)^{(4)} \Big|_{\gamma=1}, \tag{20}$$

with a similar formula for hard-core bosons, where

$$H_P^{(2+a)} \Big|_{\substack{\delta'=0 \\ \epsilon'=\omega'=\beta_a}} = \frac{\beta_a}{8} \left(\sum_{\langle x,y \rangle_{1,h}} s_x s_y - \sum_{\langle x,y \rangle_{1,v}} s_x s_y - \sum_{\langle x,y \rangle_2} s_x s_y \right). \tag{21}$$

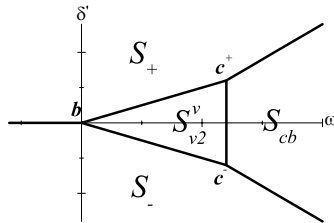


Fig. 8. The case of an intermediate anisotropy ($0 < a < 2$), off the hole-particle symmetry, with $\epsilon' = 0$. The phase diagram (common for hopping fermion and hardcore boson systems) of $H_P^{(2+a)}$. In the (ω', δ') -plane, **b** = $(0, 0)$, **c**⁺ = $(2\beta_a, \beta_a)$, **c**⁻ = $(2\beta_a, -\beta_a)$. The boundary lines of S_- , from left to right, are: $\delta' = 0$, $\delta' = -\frac{\omega'}{2}$ and $\delta' = -\omega' + \beta_a$. The boundary lines of S_+ are obtained by changing $\delta' \rightarrow -\delta'$.

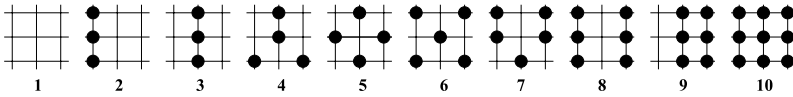


Fig. 9. All the admissible T -plaquette configurations, up to rotations by π , that are used in constructing the fourth-order phase diagram in a neighborhood of point **a** of Fig. 7, i.e. the elements of $\mathcal{S}_T^{\mathbf{a}}$.

In the previous order, it has been established that the only plaquette configurations minimizing $H_P^{(2+a)}|_{\substack{\delta'=0 \\ \varepsilon'=\omega'=\beta_a}}$ are the ones obtained by restricting to a plaquette P the periodic configurations \mathcal{S}_+ , \mathcal{S}_- , \mathcal{S}_{cb} , and \mathcal{S}_{v2}^v . Let us denote this set of plaquette configurations by $\mathcal{S}_P^{\mathbf{a}}$. Consequently, the minimization of the fourth-order potentials $(H_T^f)^{(4)}|_{\gamma=1}$, $(H_T^b)^{(4)}|_{\gamma=1}$, should be carried out only over the set $\mathcal{S}_T^{\mathbf{a}}$ of those T -plaquette configurations whose restriction to a plaquette P belongs to $\mathcal{S}_P^{\mathbf{a}}$ (Fig. 9). It appears that on the set $\mathcal{S}_T^{\mathbf{a}}$, the potentials $(H_T^f)^{(4)}|_{\gamma=1}$, $(H_T^b)^{(4)}|_{\gamma=1}$, are m -potentials. The obtained phase diagrams, independent of the anisotropy parameter β_a , are shown in Figs. 10 and 11.

B. A neighborhood of **b**

Here the system is not hole-particle symmetric and $\varepsilon' = 0$. A convenient change of variables is:

$$\omega' = t^{2-a}\omega, \quad \delta' = t^{2-a}\delta. \tag{22}$$

Then, the fourth-order effective Hamiltonian assumes the form:

$$(E_S^f)^{(4)} = \frac{t^{2+a}}{2} \sum_P H_P^{(2+a)}|_{\substack{\delta'=0 \\ \varepsilon'=\omega'=0}} + \frac{t^4}{2} \sum_T (H_T^f)^{(4)}|_{\gamma=1}, \tag{23}$$

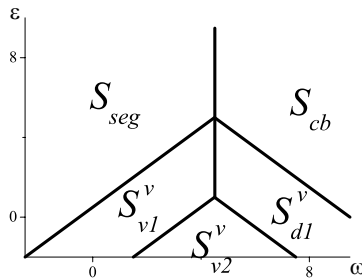


Fig. 10. The case of an intermediate anisotropy ($0 < a < 2$) and of hole-particle symmetric systems ($\mu = 0$). The phase diagram of $(E_S^f)^{(4)}$ in a neighborhood of point **a** ($\omega' = \beta_a$, $\varepsilon' = \beta_a$) (see formula (20)). The equations defining the boundary lines of the phase domains are given in Table VI of Appendix B.

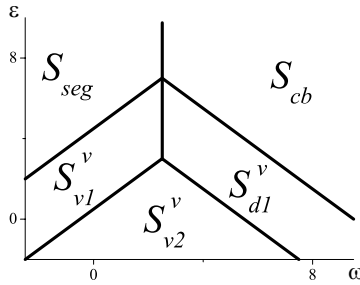


Fig. 11. The case of an intermediate anisotropy ($0 < a < 2$) and of hole-particle symmetric systems ($\mu = 0$). The phase diagram of $(E_S^b)^{(4)}$ in a neighborhood of point **a** ($\omega' = \beta_a, \epsilon' = \beta_a$) (see formula (20)). The equations defining the boundary lines of the phase domains are given in Table VI of Appendix B.

where

$$H_P^{(2+a)} \Big|_{\substack{\delta'=0 \\ \epsilon'=\omega'=0}} = -\frac{\beta_a}{4} \sum'_{\langle x,y \rangle_{1,v}} s_x s_y, \tag{24}$$

with a similar formula for hard-core bosons. The minimum of $H_P^{(2+a)} \Big|_{\substack{\delta'=0 \\ \epsilon'=\omega'=0}}$ is attained at the configurations belonging to \mathcal{S}_P^b , i.e. the plaquette configurations obtained by restricting the periodic configurations $\mathcal{S}_+, \mathcal{S}_-,$ and \mathcal{S}_{v2}^v to a plaquette P . Let \mathcal{S}_T^b be the corresponding set of T -plaquette configurations (there are no vertical pairs of n.n. sites occupied by one ion, Fig. 12). Here the potentials $(H_T^f)^{(4)}|_{\gamma=1}$ are not the m -potentials. The obtained phase diagrams are shown in Figs. 13 and 14.

C. A neighborhood of c^-

Here the system is not hole-particle symmetric and $\epsilon' = 0$. A convenient change of variables is:

$$\omega' = 2\beta_a + t^{2-a}\omega, \quad \delta' = -\beta_a + t^{2-a}\delta. \tag{25}$$

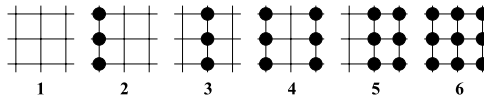


Fig. 12. All the admissible T -plaquette configurations that are used in constructing the fourth-order phase diagram, up to rotations by π , in a neighborhood of point **b** of Fig. 8, i.e. the elements of \mathcal{S}_T^b (the vertical n.n. pairs that are occupied by one ion are forbidden).

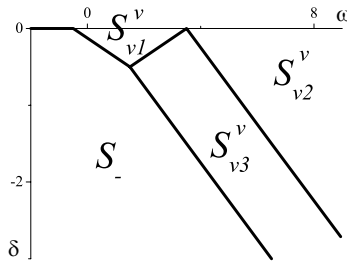


Fig. 13. The case of an intermediate anisotropy ($0 < a < 2$), off the hole-particle symmetry, with $\varepsilon' = 0$. The phase diagram of $(E_S^f)^{(4)}$ in a neighborhood of point **b** ($\omega' = 0, \delta' = 0$), see formula (23). The equations defining the boundary lines of the phase domains are given in Table VII of Appendix B, while the corresponding zero potentials—in Appendix A and in Table XXVI of Appendix C.

The fourth-order effective Hamiltonian reads:

$$(E_S^f)^{(4)} = \frac{t^{2+a}}{2} \sum_P H_P^{(2+a)} \Big|_{\substack{\varepsilon'=0 \\ \delta'=-\beta_a, \omega'=2\beta_a}} + \frac{t^4}{2} \sum_T (H_T^f)^{(4)} \Big|_{\gamma=1}, \quad (26)$$

with a similar formula for hardcore bosons, where

$$H_P^{(2+a)} \Big|_{\substack{\varepsilon'=0 \\ \delta'=-\beta_a, \omega'=2\beta_a}} = \frac{\beta_a}{4} \sum'_{\langle x,y \rangle_{1,h}} (s_x + s_y + s_x s_y + 1). \quad (27)$$

The potentials $H_P^{(2+a)} \Big|_{\substack{\varepsilon'=0 \\ \delta'=-\beta_a, \omega'=2\beta_a}}$ are minimized by the restrictions to a plaquette P of periodic configurations $S_-, S_{cb},$ and S_{v2}^v , that constitute the set S_P^c . The corresponding set S_T^c of T -plaquette configurations consists of configurations where no horizontal pair of n.n. sites is occupied by two ions (Fig. 15).

Here the potentials $(H_T^f)^{(4)} \Big|_{\gamma=1}$ are not the m -potentials. The corresponding phase diagrams are shown in Figs. 16 and 17.

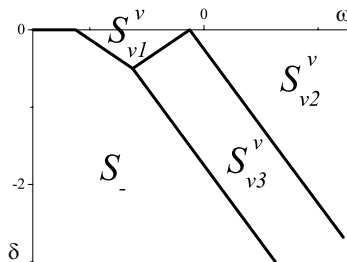


Fig. 14. The case of an intermediate anisotropy ($0 < a < 2$), off the hole-particle symmetry, with $\varepsilon' = 0$. The phase diagram of $(E_S^b)^{(4)}$ in a neighborhood of point **b** ($\omega' = 0, \delta' = 0$), see formula (23). The equations defining the boundary lines of the phase domains are given in Table VII of Appendix B, while the corresponding zero potentials—in Appendix A and in Table XXVII of Appendix C.

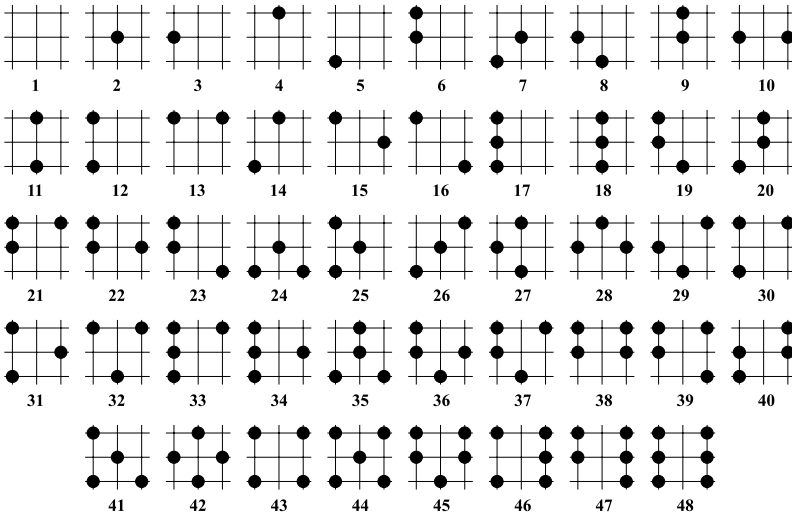


Fig. 15. All the admissible T -plaquette configurations, up to rotations by π and reflections in lattice lines parallel to the axes, that are used in constructing the fourth-order phase diagram in a neighborhood of point \mathbf{c}^- of Fig. 8, i.e. the elements of $\mathcal{S}_T^{\mathbf{c}^-}$ (the horizontal n.n. pairs that are occupied by two ions are forbidden).

4. DISCUSSION OF THE PHASE DIAGRAMS AND CONCLUSIONS

In SubSec. 3.2, we have obtained phase diagrams according to the truncated effective Hamiltonians of fourth order, with an intermediate anisotropy of hopping. These diagrams have been constructed to see what happens to the fourth-order phase diagrams with the weakest anisotropy if the anisotropy parameter β_2 grows beyond the values taken into account in SubSec. 3.1. First, let us consider the

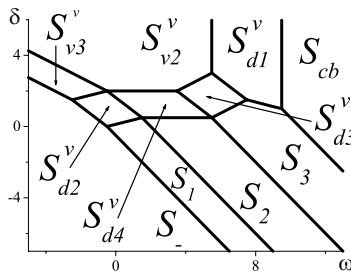


Fig. 16. The case of an intermediate anisotropy ($0 < a < 2$), off the hole-particle symmetry, with $\varepsilon' = 0$. The phase diagram of $(E_S^f)^{(4)}$ in a neighborhood of point \mathbf{c}^- ($\omega' = 2\beta_a$, $\delta' = -\beta_a$), see formula (26). The equations defining the boundary lines of the phase domains are given in Table VIII of Appendix B, while the corresponding zero potentials—in Appendix A and in Table XXVIII of Appendix C.

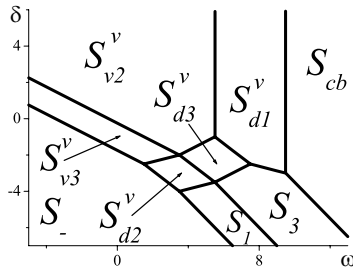


Fig. 17. The case of an intermediate anisotropy ($0 < a < 2$), off the hole-particle symmetry, with $\varepsilon' = 0$. The phase diagram of $(E_S^b)^{(4)}$ in a neighborhood of point \mathbf{c}^- ($\omega' = 2\beta_a$, $\delta' = -\beta_a$), see formula (26). The equations defining the boundary lines of the phase domains are given in Table VIII of Appendix B, while the corresponding zero potentials—in Appendix A and in Table XXIX of Appendix C.

diagrams in a neighborhood of point **a**. Apparently, up to a rescaling the phase diagram in Fig. 10 looks the same as the piece in the dotted circle of the fermionic phase diagram in Fig. 1. Similarly, up to a rescaling the phase diagram in Fig. 11 looks the same as the the piece in the dotted circle of the bosonic phase diagram in Fig. 1. The same remarks apply to the phase diagrams in the neighborhoods of points **b** and \mathbf{c}^- . The phase diagram in Fig. 13 reproduces the piece in the upper dotted ellipse in Fig. 2, while the diagram in Fig. 14—the piece in the upper dotted ellipse in Fig. 4. Then, the phase diagram in Fig. 16 reproduces the piece in the lower dotted ellipse in Fig. 2, while the diagram in Fig. 17—the piece in the lower dotted ellipse in Fig. 4. Therefore, we conclude that in the case of the weakest anisotropy in the fourth-order effective Hamiltonians, we have determined all the critical values of the anisotropy parameter β_2 . For both kinds of hopping particles, if β_2 exceeds the greatest critical value, the obtained phase diagram undergoes only a translation (varying with β_2).

Now, the basic question to be answered is concerned with the relation between these fourth-order phase diagrams and the phase diagrams of quantum systems described by Hamiltonians H_0 . By adapting the arguments presented in Refs. 20, 22, we can demonstrate, see for instance Ref. 23, that if the remainders, $(R_S^f)^{(4)}$ and $(R_S^b)^{(4)}$, are taken into account, then there is a sufficiently small t_0 such that for $t < t_0$ the phase diagrams of quantum systems look the same as the phase diagrams according to the effective Hamiltonians truncated at the fourth order, except some narrow regions, of width $O(t^2)$ (at the diagrams displayed above), located along the phase-domains boundaries, and except the domains S_{d2}^v and S_{d4}^v . For $t < t_0$ and for each domain S_D , which is different from S_{d2}^v and S_{d4}^v , there is a nonempty two-dimensional open domain S_D^∞ that is contained in the domain S_D and such that in S_D^∞ the set of ground-state configurations coincides with S_D . Moreover, in comparison with the critical values of the anisotropy parameter β_2 , determined

according to the fourth-order effective Hamiltonians, the corresponding critical values of the quantum systems described by Hamiltonians H_0 differ by $O(t^2)$, i.e. for a quantum system $\gamma = 1 - \beta_2 t^2 + O(t^4)$.

Remarkably, in the fourth-order the hole-particle symmetric phase diagrams of fermions and of hardcore bosons are geometrically similar. That is, a phase diagram of hard core bosons, with any $\beta_2 \geq 0$, can be obtained from a phase diagram of fermions, with $\beta_2 \geq 5$, by the translation whose vector reads: $\omega = -7$, $\varepsilon = -3$, and $\beta_2 = -5$. The existence of this translation vector is related to the fact that for both kinds of systems there is only one critical value of the anisotropy parameter. Additionally, for fermions with $\beta_2 < 7$, it is necessary to replace the phase \mathcal{S}_{dd} in the central domain by \mathcal{S}_{pcb} . Off the hole-particle symmetry, the relation between the bosonic and fermionic phase diagrams is not that close. For fermions, there are three critical values of β_2 , while for hardcore bosons there are no critical values. Thus, the system with hopping hardcore bosons is less sensitive to the anisotropy of hopping, than the system with hopping fermions. Nevertheless, if $\varepsilon = 0$, then the phase diagrams of both systems are topologically similar, except that in the bosonic phase diagrams the phases \mathcal{S}_{d3}^v and \mathcal{S}_2 are missing. However, we know from Ref. 23 that this deficiency can be removed by switching on the n.n.n. interactions with negative ε .

In the fourth-order effective Hamiltonians (8), the weakest anisotropy of n.n. hopping assumes the form of a fourth-order attractive n.n. interaction in vertical direction (i.e. the direction of the weaker hopping). This interaction favors n.n. pairs of occupied or empty sites which are oriented vertically. As a result, the dimeric and axial-stripe phases oriented vertically are stabilized for any value of the anisotropy parameter β_2 , while \mathcal{S}_{pcb} and \mathcal{S}_{dd} are replaced by \mathcal{S}_{v2}^v above a critical value of β_2 . Note however, that at any higher order $2k$, $k = 3, 4, \dots$, the weakest anisotropy of n.n. hopping will cause the same effects, in the effective Hamiltonians as well as in the corresponding phase diagrams. This implies that in the quantum systems described by H_0 , an arbitrary small anisotropy of n.n. hopping orients the dimeric and axial-stripe phases in the direction of the weaker hopping.

5. SUMMARY

We have considered two systems of correlated quantum particles, described by extended Falicov-Kimball Hamiltonians, with the hopping particles being either spinless fermions or hardcore bosons. Both system have been studied in the regime of strong coupling and half-filling, where the stability of some charge-stripe phases can be proved.⁽¹⁾ Two main conclusions have been drawn. First, an arbitrary small anisotropy of nearest-neighbor hopping orients the dimeric and axial-stripe phases in the direction of the weaker hopping. Second, even a weak anisotropy of hopping reveals a tendency of fermionic phase diagrams to become similar to the bosonic ones.

APPENDIX A

Here we present some details pertaining to the m -potential method, which is used in this paper to construct ground-state phase diagrams.

In comparison to the isotropic case, the presence of the hopping anisotropy lowers the symmetry of the Hamiltonians. Therefore, the zero-potentials $K_T^{(4)}$ and the set of T -plaquette configurations have to be modified suitably. The zero-potentials, satisfying

$$\sum_T K_T^{(4)} = 0, \quad (28)$$

can be chosen in the form:

$$K_T^{(4)} = \sum_{i=1}^9 \alpha_i k_T^{(i)}, \quad (29)$$

where the coefficients α_i have to be determined in the process of constructing a phase diagram, and the potentials $k_T^{(i)}$, invariant with respect to the spatial symmetries of H_0 and fulfilling condition (28), read:

$$k_T^{(1)} = s_1 + s_3 + s_7 + s_9 - 4s_5,$$

$$k_T^{(2)} = s_2 + s_8 - 2s_5,$$

$$k_T^{(3)} = s_4 + s_6 - 2s_5,$$

$$k_T^{(4)} = s_1s_2 + s_2s_3 + s_7s_8 + s_8s_9 - 2s_4s_5 - 2s_5s_6,$$

$$k_T^{(5)} = s_1s_4 + s_3s_6 + s_4s_7 + s_6s_9 - 2s_2s_5 - 2s_5s_8,$$

$$k_T^{(6)} = s_1s_5 + s_3s_5 + s_5s_9 + s_5s_7 - s_2s_4 - s_4s_8 - s_8s_6 - s_2s_6,$$

$$k_T^{(7)} = s_1s_3 + s_7s_9 - 2s_4s_6,$$

$$k_T^{(8)} = s_1s_7 + s_3s_9 - 2s_2s_8,$$

$$k_T^{(9)} = s_1s_2s_4 + s_6s_8s_9 + s_2s_3s_6 + s_4s_7s_8$$

$$- s_2s_4s_5 - s_5s_6s_8 - s_2s_5s_6 - s_4s_5s_8.$$

In the above expressions the sites of a T -plaquette have been labeled $1, \dots, 9$, from left to right, starting at the bottom left corner and ending in the upper right one. In comparison with the isotropic case, the set of T -plaquette configurations is considerably larger and consists of 168 configurations (see Fig. 3).

It turns up, that the coefficients α_i can be chosen as affine functions of the parameters that enter linearly into the truncated effective Hamiltonian, i.e. energy parameters and the anisotropy parameter. Then, the energies of T -plaquette configurations become affine functions of these parameters. Consequently, the

phase domains, being the solutions of a finite number of weak inequalities between the T -plaquette energies, are polyhedral convex sets in the space of energy and anisotropy parameters. Any point of a polyhedron is a convex combination of a finite number of points and directions (generating points and directions).⁽²⁴⁾ The coefficients α_i , determined at the generating points and half-lines being boundaries of unbounded domains, enable one to find their values at any other point by taking suitable convex combinations. This is the content of the tables presented in Appendix C.

APPENDIX B

Here we provide some details referring to the presented phase diagrams: equations of line boundaries between the phase domains, and coordinates of the crossing points of the line boundaries. The symbols like $\mathcal{S}_1|\mathcal{S}_2$ stand for the line boundary between the phases \mathcal{S}_1 and \mathcal{S}_2 , etc.

There are two kinds of entries: a formula or —, where — means that the corresponding line or a point does not exist in the phase diagram. Each entry is located in the center of columns (this center can be in a column or in between) to which it applies.

Table II. Domain boundaries of the phase diagrams for fermions, shown in Fig. 1.

	$0 < \beta_2 \leq 7$	$7 \leq \beta_2 \leq 10$
Lines		
$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\omega = \beta_2 + \frac{9}{2}$	—
$\mathcal{S}_{seg} \mathcal{S}_{dd}$	$\varepsilon = 2\omega - 2\beta_2 + 3$	—
$\mathcal{S}_{dd} \mathcal{S}_{cb}$	$\varepsilon = -2\omega + 2\beta_2 + 21$	—
$\mathcal{S}_{seg} \mathcal{S}_{v1}^v$	$\varepsilon = \omega + \frac{1}{2}$	—
$\mathcal{S}_{d1}^v \mathcal{S}_{cb}$	$\varepsilon = -\omega + 2\beta_2 + \frac{19}{2}$	—
$\mathcal{S}_{v1}^v \mathcal{S}_{dd}$	$\omega = 2\beta_2 - \frac{5}{2}$	—
$\mathcal{S}_{dd} \mathcal{S}_{d1}^v$	$\omega = \frac{23}{2}$	—
$\mathcal{S}_{v2}^v \mathcal{S}_{dd}$	$\varepsilon = 2\beta_2 - 6$	—
$\mathcal{S}_{v1}^v \mathcal{S}_{v2}^v$	$\varepsilon = \omega - \frac{7}{2}$	—
$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$	$\varepsilon = -\omega + 2\beta_2 + \frac{11}{2}$	—
$\mathcal{S}_{v1}^v \mathcal{S}_{d1}^v$	—	$\omega = \beta_2 + \frac{9}{2}$
Points		
a	$(\beta_2 + \frac{9}{2}, 12)$	$(\beta_2 + \frac{9}{2}, \beta_2 + 5)$
b	$(2\beta_2 - \frac{5}{2}, 2\beta_2 - 2)$	$(\beta_2 + \frac{9}{2}, \beta_2 + 1)$
c	$(\frac{23}{2}, 2\beta_2 - 2)$	—
d	$(2\beta_2 - \frac{5}{2}, 2\beta_2 - 6)$	—
e	$(\frac{23}{2}, 2\beta_2 - 6)$	—

Table III. Domain boundaries of the phase diagrams for hard-core bosons, shown in Fig. 1.

	$0 < \beta_2 \leq 2$	$2 < \beta_2 \leq 5$
Lines		
$\mathcal{S}_{seg} \mathcal{S}_{cb}$		$\omega = \beta_2 + \frac{5}{2}$
$\mathcal{S}_{seg} \mathcal{S}_{pcb}$	$\varepsilon = 2\omega - 2\beta_2 + 4$	—
$\mathcal{S}_{pcb} \mathcal{S}_{cb}$	$\varepsilon = -2\omega + 2\beta_2 + 14$	—
$\mathcal{S}_{seg} \mathcal{S}_{v1}^v$		$\varepsilon = \omega + \frac{9}{2}$
$\mathcal{S}_{d1}^v \mathcal{S}_{cb}$		$\varepsilon = -\omega + 2\beta_2 + \frac{19}{2}$
$\mathcal{S}_{v1}^v \mathcal{S}_{pcb}$	$\omega = 2\beta_2 + \frac{1}{2}$	—
$\mathcal{S}_{pcb} \mathcal{S}_{d1}^v$	$\omega = \frac{9}{2}$	—
$\mathcal{S}_{v2}^v \mathcal{S}_{pcb}$	$\varepsilon = 2\beta_2 + 1$	—
$\mathcal{S}_{v1}^v \mathcal{S}_{v2}^v$		$\varepsilon = \omega + \frac{1}{2}$
$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$		$\varepsilon = -\omega + 2\beta_2 + \frac{11}{2}$
$\mathcal{S}_{v1}^v \mathcal{S}_{d1}^v$	—	$\omega = \beta_2 + \frac{5}{2}$
Points		
a	$(\beta_2 + \frac{5}{2}, 9)$	$(\beta_2 + \frac{5}{2}, \beta_2 + 7)$
b	$(2\beta_2 + \frac{1}{2}, 2\beta_2 + 5)$	$(\beta_2 + \frac{5}{2}, \beta_2 + 3)$
c	$(\frac{9}{2}, 2\beta_2 + 5)$	—
d	$(2\beta_2 + \frac{1}{2}, 2\beta_2 + 1)$	—
e	$(\frac{9}{2}, 2\beta_2 + 1)$	—

Table IV. Domain boundaries of the phase diagrams shown in Fig. 2.

	$0 < \beta_2 \leq 1$	$1 < \beta_2 \leq 2$	$2 < \beta_2 \leq 3$	$3 < \beta_2 \leq 6$
Lines				
$\mathcal{S}_+ \mathcal{S}_-$		$\delta = 0$		
$\mathcal{S}_{dd} \mathcal{S}_-$	$\delta = -\frac{\omega}{2} + \frac{\beta_2}{2} - \frac{3}{4}$			—
$\mathcal{S}_{dd} \mathcal{S}_1$	$\delta = -\frac{\omega}{6} + \frac{\beta_2}{6} - \frac{11}{12}$			—
$\mathcal{S}_{dd} \mathcal{S}_2$	$\delta = -\frac{3}{2}$			—
$\mathcal{S}_{dd} \mathcal{S}_3$	$\delta = \frac{\omega}{2} - \frac{\beta_2}{2} - \frac{21}{4}$			—
$\mathcal{S}_- \mathcal{S}_1$		$\delta = -\omega + \beta_2 - \frac{1}{2}$		
$\mathcal{S}_1 \mathcal{S}_2$		$\delta = -\omega + \beta_2 + 2$		
$\mathcal{S}_2 \mathcal{S}_3$		$\delta = -\omega + \beta_2 + 6$		
$\mathcal{S}_3 \mathcal{S}_{cb}$		$\delta = -\omega + \beta_2 + \frac{21}{2}$		
$\mathcal{S}_- \mathcal{S}_{v1}^v$	—		$\delta = -\frac{\omega}{4} - \frac{1}{8}$	
$\mathcal{S}_{d2}^v \mathcal{S}_{dd}$	—	$\delta = -\frac{\omega}{4} + \frac{\beta_2}{2} - \frac{9}{8}$		—

Table IV. Continued.

	$0 < \beta_2 \leq 1$	$1 < \beta_2 \leq 2$	$2 < \beta_2 \leq 3$	$3 < \beta_2 \leq 6$
$S_{d2}^v S_-$	—		$\delta = -\frac{3\omega}{4} + \frac{\beta_2}{2} - \frac{3}{8}$	
$S_{d2}^v S_1$	—		$\delta = \frac{\omega}{4} - \frac{3\beta_2}{2} + \frac{1}{8}$	
$S_{d1}^v S_3$	—		$\delta = -\frac{\omega}{4} - \frac{\beta_2}{2} + \frac{27}{8}$	
$S_{v1}^v S_{dd}$	—		$\omega = 2\beta_2 - \frac{5}{2}$	—
$S_{d1}^v S_{dd}$	—		$\omega = \frac{23}{2}$	—
$S_{d1}^v S_{cb}$	—		$\omega = 2\beta_2 + \frac{19}{2}$	
$S_{v3}^v S_{v1}^v$	—		$\delta = \frac{\omega}{4} - \frac{7}{8}$	
$S_{v3}^v S_{dd}$	—		$\delta = -\frac{\omega}{2} + \frac{3\beta_2}{2} - \frac{11}{4}$	—
$S_{v3}^v S_-$	—		$\delta = -\frac{\omega}{2} + \frac{1}{4}$	
$S_{v3}^v S_{d2}^v$	—		$\delta = \frac{\omega}{4} - \frac{3\beta_2}{2} + \frac{17}{8}$	
$S_{d4}^v S_{d2}^v$	—		$\delta = -\frac{3\omega}{4} + \frac{\beta_2}{2} + \frac{13}{8}$	
$S_{d4}^v S_{dd}$	—		$\delta = \frac{\beta_2}{2} - \frac{5}{2}$	—
$S_{d4}^v S_2$	—		$\delta = -\beta_2 + \frac{1}{2}$	
$S_{d4}^v S_{d3}^v$	—		$\delta = -\frac{3\omega}{4} + \frac{\beta_2}{2} + \frac{37}{8}$	
$S_{d3}^v S_{dd}$	—		$\delta = \frac{\omega}{2} + \frac{\beta_2}{2} - \frac{29}{4}$	—
$S_{d3}^v S_3$	—		$\delta = \frac{\omega}{2} - 2\beta_2 - \frac{9}{4}$	
$S_{d3}^v S_{d1}^v$	—		$\delta = -\frac{3\omega}{4} + \frac{\beta_2}{2} + \frac{57}{8}$	
$S_{v2}^v S_{v3}^v$	—	—		$\delta = -\frac{\omega}{2} + \frac{7}{4}$
$S_{v2}^v S_{d4}^v$	—	—		$\delta = -\beta_2 + 2$
$S_{v2}^v S_{d3}^v$	—	—		$\delta = \frac{\omega}{2} - 2\beta_2 + \frac{1}{4}$
$S_{v2}^v S_{d1}^v$	—	—		$\omega = 2\beta_2 + \frac{11}{2}$
Points				
a	$(\beta_2 - \frac{3}{2}, 0)$		$(-\frac{1}{2}, 0)$	
b	$(\beta_2 + \frac{1}{2}, -1)$	$(2\beta_2 - \frac{5}{2}, -\frac{\beta_2}{2} + \frac{1}{2})$	$(2\beta_2 - \frac{5}{2}, \frac{\beta_2}{2} - \frac{3}{2})$	$(\frac{7}{2}, 0)$
c	$(\beta_2 + \frac{7}{2}, -\frac{3}{2})$	$(\frac{23}{2}, -\frac{\beta_2}{2} + \frac{1}{2})$	$(\frac{23}{2}, \frac{\beta_2}{2} - \frac{3}{2})$	$(\frac{3}{2}, -\frac{1}{2})$
d	$(\beta_2 + \frac{15}{2}, -\frac{3}{2})$	$(2\beta_2 + \frac{19}{2}, -\beta_2 + 1)$	$(\frac{3}{2}, -\frac{1}{2})$	$(2\beta_2 + \frac{11}{2}, -\beta_2 + 3)$
e	$(\beta_2 + \frac{21}{2}, 0)$	$(\frac{3}{2}, \frac{\beta_2}{2} - \frac{3}{2})$	$(4\beta_2 - \frac{13}{2}, -\frac{\beta_2}{2} + \frac{1}{2})$	$(2\beta_2 - \frac{1}{2}, -\beta_2 + 2)$
f	—	$(4\beta_2 - \frac{5}{2}, -\frac{\beta_2}{2} - \frac{1}{2})$	$(2\beta_2 - \frac{5}{2}, -\beta_2 + \frac{3}{2})$	$(2\beta_2 + \frac{7}{2}, -\beta_2 + 2)$
g	—	$(2\beta_2 - \frac{1}{2}, -\beta_2)$	$(2\beta_2 + \frac{15}{2}, -\beta_2 + \frac{3}{2})$	$(2\beta_2 - \frac{5}{2}, -\beta_2 + \frac{3}{2})$
h	—	$(\beta_2 + \frac{7}{2}, -\frac{3}{2})$	$(\frac{11}{2}, \frac{\beta_2}{2} - \frac{5}{2})$	$(2\beta_2 + \frac{15}{2}, -\beta_2 + \frac{3}{2})$
i	—	$(\beta_2 + \frac{15}{2}, -\frac{3}{2})$	$(\frac{19}{2}, \frac{\beta_2}{2} - \frac{5}{2})$	$(2\beta_2 + \frac{19}{2}, -\beta_2 + 1)$
j	—	—	$(2\beta_2 + \frac{19}{2}, -\beta_2 + 1)$	$(2\beta_2 + \frac{3}{2}, -\beta_2 + \frac{1}{2})$
k	—	—	$(2\beta_2 + \frac{3}{2}, -\beta_2 + \frac{1}{2})$	$(2\beta_2 + \frac{11}{2}, -\beta_2 + \frac{1}{2})$
l	—	—	$(2\beta_2 + \frac{11}{2}, -\beta_2 + \frac{1}{2})$	$(2\beta_2 - \frac{1}{2}, -\beta_2)$
m	—	—	$(2\beta_2 - \frac{1}{2}, -\beta_2)$	—

Table V. Domain boundaries of the phase diagrams shown in Fig. 4.

Lines	$0 < \beta_2 \leq 3$	Points	$0 < \beta_2 \leq 3$
$\mathcal{S}_- \mathcal{S}_+$	$\delta = 0$	a	$(-\frac{9}{2}, 0)$
$\mathcal{S}_- \mathcal{S}_{v1}^v$	$\delta = -\frac{\omega}{4} - \frac{9}{8}$	b	$(-\frac{1}{2}, 0)$
$\mathcal{S}_{v1}^v \mathcal{S}_{v3}^v$	$\delta = \frac{\omega}{4} + \frac{1}{8}$	c	$(-\frac{5}{2}, -\frac{1}{2})$
$\mathcal{S}_{v3}^v \mathcal{S}_{v2}^v$	$\delta = -\frac{\omega}{2} - \frac{1}{4}$	d	$(2\beta_2 + \frac{11}{2}, -\beta_2 - 1)$
$\mathcal{S}_- \mathcal{S}_{v3}^v$	$\delta = -\frac{\omega}{2} - \frac{7}{4}$	e	$(2\beta_2 + \frac{7}{2}, -\beta_2 - 2)$
$\mathcal{S}_{v3}^v \mathcal{S}_{d2}^v$	$\delta = \frac{\omega}{4} - \frac{3\beta_2}{2} - \frac{23}{8}$	f	$(2\beta_2 + \frac{3}{2}, -\beta_2 - \frac{5}{2})$
$\mathcal{S}_{d2}^v \mathcal{S}_{d3}^v$	$\delta = -\frac{3\omega}{4} + \frac{\beta_2}{2} + \frac{5}{8}$	g	$(2\beta_2 + \frac{15}{2}, -\beta_2 - \frac{5}{2})$
$\mathcal{S}_- \mathcal{S}_{d2}^v$	$\delta = -\frac{3\omega}{4} + \frac{\beta_2}{2} - \frac{11}{8}$	h	$(2\beta_2 + \frac{19}{2}, -\beta_2 - 3)$
$\mathcal{S}_{d2}^v \mathcal{S}_1$	$\delta = \frac{\omega}{4} - \frac{3\beta_2}{2} - \frac{39}{8}$	i	$(2\beta_2 + \frac{11}{2}, -\beta_2 - 7/2)$
$\mathcal{S}_{v2}^v \mathcal{S}_{d3}^v$	$\delta = \frac{\omega}{2} - 2\beta_2 - \frac{15}{4}$	j	$(2\beta_2 + \frac{7}{2}, -\beta_2 - 4)$
$\mathcal{S}_{d3}^v \mathcal{S}_3$	$\delta = \frac{\omega}{2} - 2\beta_2 - \frac{25}{4}$		
$\mathcal{S}_{d3}^v \mathcal{S}_{d1}^v$	$\delta = -\frac{3\omega}{4} + \frac{\beta_2}{2} + \frac{25}{8}$		
$\mathcal{S}_{d1}^v \mathcal{S}_3$	$\delta = -\frac{\omega}{4} - \frac{\beta_2}{2} - \frac{5}{8}$		
$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$	$\omega = 2\beta_2 + \frac{11}{2}$		
$\mathcal{S}_{d1}^v \mathcal{S}_{cb}$	$\omega = 2\beta_2 + \frac{19}{2}$		
$\mathcal{S}_- \mathcal{S}_1$	$\delta = -\omega + \beta_2 - \frac{1}{2}$		
$\mathcal{S}_1 \mathcal{S}_3$	$\delta = -\omega + \beta_2 + 2$		
$\mathcal{S}_3 \mathcal{S}_{cb}$	$\delta = -\omega + \beta_2 + \frac{13}{2}$		

Table VI. Domain boundaries of the phase diagrams shown in Figs. 10 and 11.

Lines	Fermions	Bosons	Points	Fermions	Bosons
$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\omega = \frac{9}{2}$	$\omega = \frac{5}{2}$	a	$(\frac{9}{2}, 5)$	$(\frac{5}{2}, 7)$
$\mathcal{S}_{v1}^v \mathcal{S}_{d1}^v$	$\omega = \frac{9}{2}$	$\omega = \frac{5}{2}$	b	$(\frac{9}{2}, 1)$	$(\frac{5}{2}, 3)$
$\mathcal{S}_{v1}^v \mathcal{S}_{seg}$	$\varepsilon = \omega + \frac{1}{2}$	$\varepsilon = \omega + \frac{9}{2}$			
$\mathcal{S}_{d1}^v \mathcal{S}_{cb}$		$\varepsilon = -\omega + \frac{19}{2}$			
$\mathcal{S}_{v2}^v \mathcal{S}_{v1}^v$	$\varepsilon = \omega - \frac{7}{2}$	$\varepsilon = \omega + \frac{1}{2}$			
$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$		$\varepsilon = -\omega + \frac{11}{2}$			

Table VII. Domain boundaries of the phase diagrams shown in Figs. 13 and 14.

Lines	Fermions	Bosons	Points	Fermions	Bosons
$\mathcal{S}_- \mathcal{S}_+$	$\delta = 0$		a	$(-\frac{1}{2}, 0)$	$(-\frac{9}{2}, 0)$
$\mathcal{S}_{v1}^v \mathcal{S}_-$	$\delta = -\frac{\omega}{4} - \frac{1}{8}$	$\delta = -\frac{\omega}{4} - \frac{9}{8}$	b	$(\frac{7}{2}, 0)$	$(-\frac{1}{2}, 0)$
$\mathcal{S}_{v1}^v \mathcal{S}_{v3}^v$	$\delta = \frac{\omega}{4} - \frac{7}{8}$	$\delta = \frac{\omega}{4} + \frac{1}{8}$	c	$(\frac{3}{2}, -\frac{1}{2})$	$(-\frac{5}{2}, -\frac{1}{2})$
$\mathcal{S}_{v3}^v \mathcal{S}_-$	$\delta = -\frac{\omega}{2} + \frac{1}{4}$	$\delta = -\frac{\omega}{2} - \frac{7}{4}$			
$\mathcal{S}_{v3}^v \mathcal{S}_{v2}^v$	$\delta = -\frac{\omega}{2} + \frac{7}{4}$	$\delta = -\frac{\omega}{2} - \frac{1}{4}$			

Table VIII. Domain boundaries of the phase diagrams shown in Figs. 16 and 17.

Lines	Fermions	Bosons	Points	Fermions	Bosons
$\mathcal{S}_{v3}^v \mathcal{S}_-$	$\delta = -\frac{\omega}{2} + \frac{1}{4}$	$\delta = -\frac{\omega}{2} - \frac{7}{4}$	a	$(\frac{11}{2}, 3)$	$(\frac{11}{2}, -1)$
$\mathcal{S}_{v3}^v \mathcal{S}_{v2}^v$	$\delta = -\frac{\omega}{2} + \frac{7}{4}$	$\delta = -\frac{\omega}{2} - \frac{1}{4}$	b	$(-\frac{1}{2}, 2)$	$(\frac{7}{2}, -2)$
$\mathcal{S}_{v3}^v \mathcal{S}_{d2}^v$	$\delta = \frac{\omega}{4} + \frac{17}{8}$	$\delta = \frac{\omega}{4} - \frac{23}{8}$	c	$(\frac{7}{2}, 2)$	$(\frac{3}{2}, -\frac{5}{2})$
$\mathcal{S}_{d2}^v \mathcal{S}_-$	$\delta = -\frac{3\omega}{4} - \frac{3}{8}$	$\delta = -\frac{3\omega}{4} - \frac{11}{8}$	d	$(-\frac{5}{2}, \frac{3}{2})$	$(\frac{15}{2}, -\frac{5}{2})$
$\mathcal{S}_{d2}^v \mathcal{S}_{d4}^v$	$\delta = -\frac{3\omega}{4} + \frac{13}{8}$	—	e	$(\frac{15}{2}, \frac{3}{2})$	$(\frac{19}{2}, -3)$
$\mathcal{S}_{d2}^v \mathcal{S}_{d3}^v$	—	$\delta = -\frac{3\omega}{4} + \frac{5}{8}$	f	$(\frac{19}{2}, 1)$	$(\frac{11}{2}, -\frac{7}{2})$
$\mathcal{S}_{d2}^v \mathcal{S}_1$	$\delta = \frac{\omega}{4} + \frac{1}{8}$	$\delta = \frac{\omega}{4} - \frac{39}{8}$	g	$(\frac{3}{2}, \frac{1}{2})$	$(\frac{7}{2}, -4)$
$\mathcal{S}_{d4}^v \mathcal{S}_{v2}^v$	$\delta = 2$	—	h	$(\frac{11}{2}, \frac{1}{2})$	—
$\mathcal{S}_{d4}^v \mathcal{S}_2$	$\delta = \frac{1}{2}$	—			
$\mathcal{S}_{d4}^v \mathcal{S}_{d3}^v$	$\delta = -\frac{3\omega}{4} + \frac{37}{8}$	—	i	$(-\frac{1}{2}, 0)$	—
$\mathcal{S}_{d3}^v \mathcal{S}_{v2}^v$	$\delta = \frac{\omega}{2} + \frac{1}{4}$	$\delta = \frac{\omega}{2} - \frac{15}{4}$			
$\mathcal{S}_{d3}^v \mathcal{S}_3$	$\delta = \frac{\omega}{2} - \frac{9}{4}$	$\delta = \frac{\omega}{2} - \frac{25}{4}$			
$\mathcal{S}_{d3}^v \mathcal{S}_{d1}^v$	$\delta = -\frac{3\omega}{4} + \frac{57}{8}$	$\delta = -\frac{3\omega}{4} + \frac{25}{8}$			
$\mathcal{S}_{d1}^v \mathcal{S}_3$	$\delta = -\frac{\omega}{4} + \frac{27}{8}$	$\delta = -\frac{\omega}{4} - \frac{5}{8}$			
$\mathcal{S}_{d1}^v \mathcal{S}_{v2}^v$		$\omega = \frac{11}{2}$			
$\mathcal{S}_{d1}^v \mathcal{S}_{cb}$		$\omega = \frac{19}{2}$			
$\mathcal{S}_- \mathcal{S}_1$		$\delta = -\omega - \frac{1}{2}$			
$\mathcal{S}_1 \mathcal{S}_2$	$\delta = -\omega + 2$	—			
$\mathcal{S}_1 \mathcal{S}_3$	—	$\delta = -\omega + 2$			
$\mathcal{S}_2 \mathcal{S}_3$	$\delta = -\omega + 6$	—			
$\mathcal{S}_3 \mathcal{S}_{cb}$	$\delta = -\omega + \frac{21}{2}$	$\delta = -\omega + \frac{13}{2}$			

APPENDIX C

Below, in a series of tables, we provide the coefficients $\{\alpha_i\}$, $i = 1, \dots, 9$, of the zero-potentials for the phase diagrams presented in this paper. The coefficients that are missing in a table are equal to zero. As in Appendix B, the symbol $\mathcal{S}_1 | \mathcal{S}_2$ denotes the boundary between the phases \mathcal{S}_1 and \mathcal{S}_2 , etc.

Each entry is located in the center of columns (this center can be in a column or in between) to which it applies.

Table IX. Zero-potentials coefficients for the phase diagram of fermions, shown in Fig. 1, for $\beta_2 = 0$.

$\{\alpha_i\}$	$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\mathcal{S}_{seg} \mathcal{S}_{v1}$	$\mathcal{S}_{cb} \mathcal{S}_{d1}$	$\mathcal{S}_{v2} \mathcal{S}_{v1}$	$\mathcal{S}_{v2} \mathcal{S}_{d1}$
α_1			0		
α_2			0		
α_3			0		
α_4		$-\frac{\omega}{96} + \frac{3}{64}$		$-\frac{\omega}{96} + \frac{7}{64}$	$-\frac{\omega}{96} - \frac{1}{64}$
α_5		$-\frac{\omega}{96} + \frac{3}{64}$		$-\frac{\omega}{96} + \frac{7}{64}$	$-\frac{\omega}{96} - \frac{1}{64}$
α_6		0			$-\frac{1}{8}$
α_7		$-\frac{1}{48}$			$-\frac{1}{12}$
α_8		$-\frac{1}{48}$			$-\frac{1}{12}$

Table X. Zero-potentials coefficients for the phase diagram of fermions, shown in Fig. 1, for $\beta_2 = 7/2$.

$\{\alpha_i\}$	$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\mathcal{S}_{seg} \mathcal{S}_{v1}^v$	$\mathcal{S}_{cb} \mathcal{S}_{d1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{v1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$
α_1			0		
α_2			0		
α_3			0		
α_4			$-\frac{\omega}{96} + \frac{3}{64}$		
α_5			$-\frac{\omega}{96} + \frac{23}{192}$		
α_6			0		
α_7			$-\frac{1}{48}$		
α_8			$-\frac{1}{48}$		

Table XI. Zero-potentials coefficients for the phase diagram of fermions, shown in Fig. 1, for $\beta_2 = 7$.

$\{\alpha_i\}$	$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\mathcal{S}_{seg} \mathcal{S}_{v1}^v$	$\mathcal{S}_{cb} \mathcal{S}_{d1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{v1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$
α_1			0		
α_2			0		
α_3			0		
α_4		$-\frac{\omega}{96} - \frac{1}{64}$		$-\frac{\omega}{96} + \frac{3}{64}$	
α_5		$-\frac{\omega}{96} + \frac{49}{192}$		$-\frac{\omega}{96} + \frac{37}{192}$	
α_6		$\frac{1}{8}$		0	
α_7		$-\frac{1}{12}$		$-\frac{1}{48}$	
α_8		$-\frac{1}{12}$		$-\frac{1}{48}$	

Table XII. Zero-potentials coefficients for the phase diagram of fermions, shown in Fig. 1, for $\beta_2 = 10$.

$\{\alpha_i\}$	$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\mathcal{S}_{seg} \mathcal{S}_{v1}^v$	$\mathcal{S}_{cb} \mathcal{S}_{d1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{v1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$
α_1			0		
α_2			0		
α_3			0		
α_4		$-\frac{1}{6}$		$-\frac{\omega}{96} + \frac{3}{64}$	
α_5		$\frac{1}{6} + \frac{49}{192}$		$-\frac{\omega}{96} + \frac{49}{192}$	
α_6		$\frac{1}{8}$		0	
α_7		$-\frac{1}{12}$		$-\frac{1}{48}$	
α_8		$-\frac{1}{12}$		$-\frac{1}{48}$	

Table XIII. Zero-potentials coefficients for the phase diagram of hardcore bosons, shown in Fig. 1, for $\beta_2 = 0$.

$\{\alpha_i\}$	$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\mathcal{S}_{seg} \mathcal{S}_{v1}$	$\mathcal{S}_{cb} \mathcal{S}_{d1}$	$\mathcal{S}_{v2} \mathcal{S}_{v1}$	$\mathcal{S}_{v2} \mathcal{S}_{d1}$
α_1			0		
α_2			0		
α_3			0		
α_4		$-\frac{\omega}{96} + \frac{5}{192}$		$-\frac{\omega}{96} + \frac{17}{192}$	$-\frac{\omega}{96} - \frac{7}{192}$
α_5		$-\frac{\omega}{96} + \frac{5}{192}$		$-\frac{\omega}{96} + \frac{17}{192}$	$-\frac{\omega}{96} - \frac{7}{192}$
α_6		0			$-\frac{1}{8}$
α_7		$-\frac{1}{48}$			$-\frac{1}{12}$
α_8		$-\frac{1}{48}$			$-\frac{1}{12}$

Table XIV. Zero-potentials coefficients for the phase diagram of hardcore bosons, shown in Fig. 1, for $\beta_2 = 1$.

$\{\alpha_i\}$	$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\mathcal{S}_{seg} \mathcal{S}_{v1}^v$	$\mathcal{S}_{cb} \mathcal{S}_{d1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{v1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$
α_1			0		
α_2			0		
α_3			0		
α_4	$-\frac{1}{24}$			$-\frac{\omega}{96} + \frac{5}{192}$	
α_5	$\frac{1}{24}$			$-\frac{\omega}{96} + \frac{3}{64}$	
α_6			0		
α_7			$-\frac{1}{48}$		
α_8			$-\frac{1}{48}$		

Table XV. Zero-potentials coefficients for the phase diagram of hardcore bosons, shown in Fig. 1, for $\beta_2 = 2$.

$\{\alpha_i\}$	$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\mathcal{S}_{seg} \mathcal{S}_{v1}^v$	$\mathcal{S}_{cb} \mathcal{S}_{d1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{v1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$
α_1			0		
α_2			0		
α_3			0		
α_4	$-\frac{1}{12}$		$-\frac{\omega}{96} - \frac{1}{192}$		$-\frac{\omega}{96} + \frac{1}{24}$
α_5	$\frac{1}{12}$			$-\frac{\omega}{96} + \frac{19}{192}$	
α_6	$\frac{1}{8}$		$\frac{1}{32}$		$-\frac{1}{32}$
α_7	$-\frac{1}{12}$		$-\frac{1}{48}$		$-\frac{1}{192}$
α_8	$-\frac{1}{12}$		$-\frac{1}{48}$		$-\frac{5}{96}$

Table XVI. Zero-potentials coefficients for the phase diagram of hardcore bosons, shown in Fig. 1, for $\beta_2 = 5$.

$\{\alpha_i\}$	$\mathcal{S}_{seg} \mathcal{S}_{cb}$	$\mathcal{S}_{seg} \mathcal{S}_{v1}^v$	$\mathcal{S}_{cb} \mathcal{S}_{d1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{v1}^v$	$\mathcal{S}_{v2}^v \mathcal{S}_{d1}^v$
α_1			0		
α_2			0		
α_3			0		
α_4		$-\frac{11}{96}$		$-\frac{\omega}{96} + \frac{1}{24}$	
α_5		$\frac{11}{96}$		$-\frac{\omega}{96} + \frac{19}{192}$	
α_6		$\frac{1}{8}$		$-\frac{1}{32}$	
α_7		$-\frac{1}{12}$		$-\frac{1}{192}$	
α_8		$-\frac{1}{12}$		$-\frac{5}{96}$	

Table XVII. Zero-potentials coefficients for the phase diagram shown in Fig. 2, for $\beta_2 = 0$.

$\{\alpha_i\}$	$\mathcal{S}_+ \mathcal{S}_-$	$\mathcal{S}_- \mathcal{S}_1$	$\mathcal{S}_1 \mathcal{S}_2$	$\mathcal{S}_2 \mathcal{S}_3$	$\mathcal{S}_3 \mathcal{S}_{cb}$	$\delta = 0, \omega \geq \frac{21}{2}$
α_1	0	$\frac{\omega}{16} + \frac{3}{32}$	$\frac{\omega}{16} - \frac{3}{32}$	$\frac{\omega}{16} - \frac{9}{32}$	$\frac{\omega}{16} - \frac{21}{32}$	0
α_2	0	$\frac{\omega}{8} + \frac{1}{16}$	$\frac{\omega}{8} - \frac{3}{16}$	$\frac{\omega}{8} - \frac{3}{4}$	$\frac{\omega}{8} - \frac{21}{16}$	0
α_3	0	$\frac{\omega}{8} + \frac{1}{16}$	$\frac{\omega}{8} - \frac{3}{16}$	$\frac{\omega}{8} - \frac{3}{4}$	$\frac{\omega}{8} - \frac{21}{16}$	0
α_4		$-\frac{\omega}{96} + \frac{3}{64}$		$-\frac{\omega}{96} + \frac{1}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	
α_5		$-\frac{\omega}{96} + \frac{3}{64}$		$-\frac{\omega}{96} + \frac{1}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	
α_6		0	$-\frac{1}{16}$	$-\frac{1}{32}$		0
α_7		$-\frac{1}{48}$		$\frac{1}{96}$		$-\frac{1}{48}$
α_8		$-\frac{1}{48}$		$\frac{1}{96}$		$-\frac{1}{48}$

Table XVIII. Zero-potentials coefficients for the phase diagram shown in Fig. 2, for $\beta_2 = 1$.

$\{\alpha_i\}$	$\mathcal{S}_+ \mathcal{S}_-$	$\mathcal{S}_- \mathcal{S}_1$	$\mathcal{S}_1 \mathcal{S}_2$	$\mathcal{S}_2 \mathcal{S}_3$	$\mathcal{S}_3 \mathcal{S}_{cb}$	$\delta = 0, \omega \geq \frac{23}{2}$
α_1	0	$\frac{\omega}{16} + \frac{1}{32}$	$\frac{\omega}{16} - \frac{3}{32}$	$\frac{\omega}{16} - \frac{11}{32}$	$\frac{\omega}{16} - \frac{23}{32}$	0
α_2	0	$\frac{\omega}{8} - \frac{1}{16}$	$\frac{\omega}{8} - \frac{11}{32}$	$\frac{\omega}{8} - \frac{41}{48}$	$\frac{\omega}{8} - \frac{23}{16}$	0
α_3	0	$\frac{\omega}{8} - \frac{1}{16}$	$\frac{\omega}{8} - \frac{13}{32}$	$\frac{\omega}{8} - \frac{43}{48}$	$\frac{\omega}{8} - \frac{23}{16}$	0
α_4		$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{9}{128}$	$-\frac{\omega}{96} + \frac{13}{192}$	$-\frac{\omega}{96} + \frac{13}{64}$	
α_5		$-\frac{\omega}{96} + \frac{13}{192}$	$-\frac{\omega}{96} + \frac{29}{384}$	$-\frac{\omega}{96} + \frac{5}{64}$	$-\frac{\omega}{96} + \frac{13}{192}$	
α_6		0		$-\frac{1}{32}$		0
α_7		$-\frac{1}{48}$	$\frac{1}{96}$	$-\frac{1}{192}$		$-\frac{1}{48}$
α_8		$-\frac{1}{48}$	$\frac{1}{96}$	$\frac{1}{192}$		$-\frac{1}{48}$

Table XIX. Zero-potentials coefficients for the phase diagram shown in Fig. 2, for $\beta_2 = 3/2$.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
$S_+ S_-$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{5}{64}$	$\frac{1}{64}$	$-\frac{7}{192}$	$-\frac{1}{48}$	0
$S_- S_1$	$\frac{\omega}{16}$	$\frac{\omega}{8} - \frac{1}{8}$	$\frac{\omega}{8} - \frac{3}{32}$	$-\frac{\omega}{96} + \frac{15}{256}$	$-\frac{\omega}{96} + \frac{5}{64}$	$-\frac{1}{256}$	$-\frac{17}{384}$	$-\frac{1}{192}$	0
$S_1 S_2$	$\frac{\omega}{16} - \frac{49}{256}$	$\frac{\omega}{8} - \frac{31}{64}$	$\frac{\omega}{8} - \frac{13}{32}$	$-\frac{\omega}{96} + \frac{7}{128}$	$-\frac{\omega}{96} + \frac{27}{256}$	$-\frac{7}{256}$	$-\frac{23}{384}$	$-\frac{23}{384}$	$\frac{19}{256}$
$S_2 S_3$	$\frac{\omega}{16} - \frac{105}{256}$	$\frac{\omega}{8} - \frac{115}{128}$	$\frac{\omega}{8} - \frac{29}{32}$	$-\frac{\omega}{96} + \frac{7}{128}$	$-\frac{\omega}{96} + \frac{43}{512}$	$-\frac{25}{512}$	$-\frac{157}{3072}$	$\frac{113}{3072}$	0
$S_3 S_{cb}$	$\frac{\omega}{16} - \frac{3}{4}$	$\frac{\omega}{8} - \frac{3}{2}$	$\frac{\omega}{8} - \frac{3}{2}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{5}{64}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
$\delta = 0, \omega \geq \frac{25}{2}$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{5}{64}$	$\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{7}{192}$	0
$(\frac{1}{2}, 0)$	0	0	0	$\frac{1}{24}$	$\frac{7}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
$(\frac{23}{2}, 0)$	0	0	0	$-\frac{7}{96}$	$-\frac{1}{24}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
b	$\frac{1}{32}$	0	$\frac{1}{16}$	$\frac{1}{24}$	$\frac{7}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
c	$\frac{1}{32}$	$\frac{1}{16}$	0	$-\frac{7}{96}$	$-\frac{1}{24}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
e	$\frac{3}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{16}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
f	$\frac{7}{64}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{96}$	$\frac{11}{192}$	$-\frac{1}{64}$	$-\frac{5}{96}$	$-\frac{5}{96}$	$\frac{3}{64}$

Table XX. Zero-potentials coefficients for the phase diagram shown in Fig. 2, for $\beta_2 = 2$.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
$S_+ S_-$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{17}{192}$	$\frac{1}{32}$	$-\frac{5}{96}$	$-\frac{1}{48}$	0
$S_- S_1$	$\frac{\omega}{16} - \frac{1}{32}$	$\frac{\omega}{8} - \frac{3}{16}$	$\frac{\omega}{8} - \frac{3}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{17}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
$S_1 S_2$	$\frac{\omega}{16} - \frac{1}{4}$	$\frac{\omega}{8} - \frac{9}{16}$	$\frac{\omega}{8} - \frac{7}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{23}{192}$	$-\frac{1}{32}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{3}{32}$
$S_2 S_3$	$\frac{\omega}{16} - \frac{15}{32}$	$\frac{\omega}{8} - \frac{15}{16}$	$\frac{\omega}{8} - \frac{15}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{17}{192}$	$-\frac{1}{16}$	$-\frac{1}{12}$	$\frac{1}{24}$	0
$S_3 S_{cb}$	$\frac{\omega}{16} - \frac{25}{32}$	$\frac{\omega}{8} - \frac{25}{16}$	$\frac{\omega}{8} - \frac{25}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{17}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
$\delta = 0, \omega \geq \frac{27}{2}$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{17}{192}$	$\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{5}{96}$	0
$(\frac{3}{2}, 0)$	0	0	0	$\frac{1}{32}$	$\frac{7}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
$(\frac{23}{2}, 0)$	0	0	0	$-\frac{7}{96}$	$-\frac{1}{32}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
b(e)	$\frac{1}{16}$	0	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{7}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
c	$\frac{1}{16}$	$\frac{1}{8}$	0	$-\frac{7}{96}$	$-\frac{1}{32}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0

Table XXI. Zero-potentials coefficients for the phase diagram shown in Fig. 2, for $\beta_2 = 5/2$.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9
$S_+ S_-$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{19}{192}$	$\frac{1}{32}$	$-\frac{5}{96}$	$-\frac{1}{48}$	0
$S_- S_1$	$\frac{\omega}{16} - \frac{1}{16}$	$\frac{\omega}{8} - \frac{1}{4}$	$\frac{\omega}{8} - \frac{1}{4}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{19}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
$S_1 S_2$	$\frac{\omega}{16} - \frac{9}{32}$	$\frac{\omega}{8} - \frac{5}{8}$	$\frac{\omega}{8} - \frac{1}{2}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{25}{192}$	$-\frac{1}{32}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{3}{32}$
$S_2 S_3$	$\frac{\omega}{16} - \frac{33}{64}$	$\frac{\omega}{8} - \frac{31}{32}$	$\frac{\omega}{8} - 1$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{35}{384}$	$-\frac{9}{128}$	$-\frac{1}{12}$	$-\frac{5}{192}$	0
$S_3 S_{cb}$	$\frac{\omega}{16} - \frac{13}{16}$	$\frac{\omega}{8} - \frac{13}{8}$	$\frac{\omega}{8} - \frac{13}{8}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{19}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
$\delta = 0, \omega \geq \frac{29}{2}$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{19}{192}$	$\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{5}{96}$	0
$(\frac{5}{2}, 0)$	0	0	0	$\frac{1}{48}$	$\frac{7}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
$(\frac{23}{2}, 0)$	0	0	0	$-\frac{7}{96}$	$-\frac{1}{48}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
b	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{5}{96}$	0	$-\frac{3}{32}$	$-\frac{1}{12}$	$-\frac{1}{48}$	0
c	$\frac{1}{32}$	$\frac{1}{16}$	0	$-\frac{7}{96}$	$-\frac{1}{48}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
d	$\frac{1}{16}$	0	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{1}{12}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$	0
e	$\frac{1}{32}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{96}$	$\frac{1}{16}$	$-\frac{1}{16}$	$-\frac{5}{96}$	$-\frac{5}{96}$	0
f	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{7}{192}$	$\frac{7}{96}$	$-\frac{1}{64}$	$\frac{1}{96}$	$-\frac{1}{48}$	0
g	$\frac{3}{32}$	$\frac{3}{16}$	$\frac{1}{16}$	$-\frac{1}{12}$	$-\frac{3}{64}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$	0
h	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$-\frac{1}{96}$	$\frac{7}{96}$	$-\frac{1}{16}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$\frac{1}{16}$
i	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{5}{96}$	0	$-\frac{3}{32}$	$-\frac{1}{12}$	$-\frac{1}{48}$	0

Table XXII. Zero-potentials coefficients for the phase diagram shown in Fig. 2, for $\beta_2 = 3$.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$S_+ S_-$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{7}{64}$	$\frac{1}{32}$	$-\frac{5}{96}$	$-\frac{1}{48}$
$S_- S_1$	$\frac{\omega}{16} - \frac{3}{32}$	$\frac{\omega}{8} - \frac{5}{16}$	$\frac{\omega}{8} - \frac{5}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{7}{64}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_1 S_2$	$\frac{\omega}{16} - \frac{11}{32}$	$\frac{\omega}{8} - \frac{7}{16}$	$\frac{\omega}{8} - \frac{11}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{1}{16}$	$-\frac{1}{12}$	$-\frac{1}{12}$
$S_2 S_3$	$\frac{\omega}{16} - \frac{9}{16}$	$\frac{\omega}{8} - 1$	$\frac{\omega}{8} - \frac{17}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{3}{32}$	$-\frac{5}{64}$	$-\frac{1}{12}$	$\frac{1}{96}$
$S_3 S_{cb}$	$\frac{\omega}{16} - \frac{27}{32}$	$\frac{\omega}{8} - \frac{27}{16}$	$\frac{\omega}{8} - \frac{27}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{7}{64}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$\delta = 0, \omega \geq \frac{31}{2}$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{7}{64}$	$\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{5}{96}$
$(\frac{7}{2}, 0)$ (b)	0	0	0	$\frac{1}{96}$	$\frac{7}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$(\frac{23}{2}, 0)$ (c)	0	0	0	$-\frac{7}{96}$	$-\frac{1}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
d	$\frac{1}{16}$	0	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{3}{32}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
e (h)	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{96}$	$\frac{5}{96}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
i	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{5}{96}$	$\frac{1}{96}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
f	$\frac{5}{32}$	$\frac{7}{32}$	$\frac{3}{16}$	$\frac{35}{768}$	$\frac{7}{96}$	$-\frac{3}{256}$	$-\frac{5}{384}$	$-\frac{1}{48}$
g	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{3}{32}$	$-\frac{3}{64}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$

Table XXIII. Zero-potentials coefficients for the phase diagram shown in Fig. 2, for $\beta_2 = 6$.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$S_+ S_-$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{11}{64}$	$\frac{1}{32}$	$-\frac{5}{96}$	$-\frac{1}{48}$
$S_- S_1$	$\frac{\omega}{16} - \frac{9}{32}$	$\frac{\omega}{8} - \frac{11}{16}$	$\frac{\omega}{8} - \frac{11}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{11}{64}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_1 S_2$	$\frac{\omega}{16} - \frac{19}{32}$	$\frac{\omega}{8} - \frac{11}{16}$	$\frac{\omega}{8} - \frac{15}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{7}{64}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
$S_2 S_3$	$\frac{\omega}{16} - \frac{25}{32}$	$\frac{\omega}{8} - \frac{21}{16}$	$\frac{\omega}{8} - \frac{23}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{9}{64}$	$-\frac{3}{32}$	$-\frac{1}{12}$	$-\frac{1}{48}$
$S_3 S_{cb}$	$\frac{\omega}{16} - \frac{33}{32}$	$\frac{\omega}{8} - \frac{33}{16}$	$\frac{\omega}{8} - \frac{33}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{11}{64}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$\delta = 0, \omega \geq \frac{43}{2}$	0	0	0	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{\omega}{96} + \frac{11}{64}$	$\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{5}{96}$
b	0	0	0	$\frac{1}{96}$	$\frac{13}{96}$	$-\frac{1}{32}$	$-\frac{5}{96}$	$-\frac{1}{48}$
$(\frac{35}{2}, 0)$	0	0	0	$-\frac{13}{96}$	$-\frac{1}{96}$	$-\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{5}{96}$
c	$\frac{1}{16}$	0	$\frac{1}{8}$	$\frac{1}{32}$	$\frac{5}{32}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
d	$\frac{3}{16}$	$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{13}{96}$	$-\frac{5}{192}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$
e	$\frac{7}{32}$	$\frac{5}{8}$	$\frac{9}{16}$	$-\frac{11}{192}$	$\frac{5}{96}$	$-\frac{7}{64}$	$-\frac{1}{12}$	$-\frac{1}{12}$
f	$\frac{3}{16}$	$\frac{5}{8}$	$\frac{5}{8}$	$-\frac{11}{96}$	$-\frac{1}{48}$	$-\frac{3}{32}$	$-\frac{1}{12}$	$-\frac{1}{48}$
g	$\frac{11}{32}$	$\frac{5}{8}$	$\frac{9}{16}$	$-\frac{1}{192}$	$\frac{7}{96}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$
h	$\frac{5}{16}$	$\frac{5}{8}$	$\frac{1}{2}$	$-\frac{5}{32}$	$-\frac{3}{64}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$

Table XXIV. Zero-potentials coefficients for the phase diagram shown in Fig. 4, for $\beta_2 = 0$.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$S_+ S_-$	0	0	0	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{\omega}{96} + \frac{5}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_- S_1$	$\frac{\omega}{16} + \frac{3}{32}$	$\frac{\omega}{8} + \frac{1}{16}$	$\frac{\omega}{8} + \frac{1}{16}$	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{\omega}{96} + \frac{5}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_1 S_3$	$\frac{\omega}{16} - \frac{9}{32}$	$\frac{\omega}{8} - \frac{1}{16}$	$\frac{\omega}{8} - \frac{1}{16}$	$-\frac{\omega}{96} - \frac{7}{192}$	$-\frac{\omega}{96} - \frac{7}{192}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
$S_3 S_{cb}$	$\frac{\omega}{16} - \frac{13}{32}$	$\frac{\omega}{8} - \frac{13}{16}$	$\frac{\omega}{8} - \frac{13}{16}$	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{\omega}{96} + \frac{5}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$\delta = 0, \omega \geq \frac{19}{2}$	0	0	0	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{\omega}{96} + \frac{5}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
b	0	0	0	$\frac{3}{32}$	$\frac{3}{32}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
$(\frac{11}{2}, 0)$	0	0	0	$-\frac{3}{32}$	$-\frac{3}{32}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
c	$\frac{3}{32}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{12}$	$\frac{1}{12}$	$-\frac{1}{32}$	$\frac{1}{96}$	$\frac{1}{96}$
d	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{3}{32}$	$-\frac{3}{32}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
e	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{1}{96}$	$-\frac{1}{96}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
f	$\frac{3}{32}$	$\frac{9}{16}$	$\frac{9}{16}$	$\frac{7}{96}$	$\frac{7}{96}$	$-\frac{3}{32}$	$-\frac{1}{12}$	$-\frac{1}{12}$
g	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$-\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{1}{48}$

Table XXV. Zero-potentials coefficients for the phase diagram shown in Fig. 4, for $\beta_2 = 3$.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$\mathcal{S}_+ \mathcal{S}_-$	0	0	0	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{\omega}{96} + \frac{17}{192}$	$\frac{1}{32}$	$-\frac{5}{96}$	$-\frac{1}{48}$
$\mathcal{S}_- \mathcal{S}_1$	$\frac{\omega}{16} - \frac{3}{32}$	$\frac{\omega}{8} - \frac{5}{16}$	$\frac{\omega}{8} - \frac{5}{16}$	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{\omega}{96} + \frac{17}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$\mathcal{S}_1 \mathcal{S}_3$	$\frac{\omega}{16} - \frac{15}{32}$	$\frac{\omega}{8} - \frac{7}{16}$	$\frac{\omega}{8} - \frac{7}{16}$	$-\frac{\omega}{96} - \frac{7}{192}$	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
$\mathcal{S}_3 \mathcal{S}_{cb}$	$\frac{\omega}{16} - \frac{19}{32}$	$\frac{\omega}{8} - \frac{19}{16}$	$\frac{\omega}{8} - \frac{19}{16}$	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{\omega}{96} + \frac{17}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$\delta = 0, \omega \geq \frac{31}{2}$	0	0	0	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{\omega}{96} + \frac{17}{192}$	$\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{5}{96}$
b	0	0	0	$\frac{1}{32}$	$\frac{3}{32}$	$-\frac{1}{32}$	$-\frac{5}{96}$	$-\frac{1}{48}$
$(\frac{23}{2}, 0)$	0	0	0	$-\frac{3}{32}$	$-\frac{1}{32}$	$-\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{5}{96}$
c	$\frac{3}{32}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{13}{192}$	$\frac{25}{192}$	$-\frac{1}{64}$	$\frac{1}{96}$	$-\frac{1}{48}$
d	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{32}$	$-\frac{3}{64}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$
e	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$-\frac{7}{96}$	$-\frac{1}{96}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
f	$\frac{13}{32}$	$\frac{23}{32}$	$\frac{11}{16}$	$-\frac{5}{384}$	$\frac{1}{96}$	$-\frac{1}{128}$	$-\frac{1}{48}$	$-\frac{1}{48}$
g	$\frac{11}{32}$	$\frac{11}{16}$	$\frac{11}{16}$	$-\frac{7}{48}$	$-\frac{13}{192}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$

Table XXVI. Zero-potentials coefficients for the phase diagram shown in Fig. 13.

Lines	α_1
$\mathcal{S}_+ \mathcal{S}_-$	0
$\mathcal{S}_- \mathcal{S}_{v3}^v$	$\frac{\omega}{16} + \frac{1}{32}$
$\mathcal{S}_{v3}^v \mathcal{S}_{v2}^v$	$\frac{\omega}{16} - \frac{7}{32}$
$\delta = 0, \omega \geq \frac{7}{2}$	0

Table XXVII. Zero-potentials coefficients for the phase diagram shown in Fig. 14.

Lines	α_1
$\mathcal{S}_+ \mathcal{S}_-$	0
$\mathcal{S}_- \mathcal{S}_{v3}^v$	$\frac{\omega}{16} + \frac{9}{32}$
$\mathcal{S}_{v3}^v \mathcal{S}_{v2}^v$	$\frac{\omega}{16} + \frac{1}{32}$
$\delta = 0, \omega \geq -\frac{1}{2}$	0

Table XXVIII. Zero-potentials coefficients for the phase diagram shown in Fig. 16.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$S_- S_{v3}^v$	0	0	$\frac{\omega}{8} + \frac{1}{16}$	$-\frac{\omega}{24} + \frac{1}{16}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_- S_1$	0	$-\frac{1}{8}$	$\frac{\omega}{4} + \frac{1}{4}$	$-\frac{7\omega}{96} - \frac{3}{64}$	$-\frac{\omega}{96} + \frac{3}{64}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_1 S_2$	$\frac{7\omega}{96} - \frac{17}{64}$	$\frac{7\omega}{48} - \frac{1}{32}$	$\frac{5\omega}{48} - \frac{3}{32}$	0	$-\frac{\omega}{96} - \frac{1}{64}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
$S_2 S_3$	$\frac{7\omega}{96} - \frac{29}{64}$	$\frac{7\omega}{48} - \frac{21}{32}$	$\frac{5\omega}{48} - \frac{19}{32}$	0	$-\frac{\omega}{96} + \frac{1}{64}$	$-\frac{3}{32}$	$-\frac{1}{12}$	$-\frac{1}{48}$
$S_3 S_{cb}$	0	0	$\frac{\omega}{4} - \frac{21}{8}$	$-\frac{7\omega}{96} + \frac{45}{64}$	$-\frac{\omega}{96} + \frac{3}{64}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_{d1}^v S_{cb}$	0	0	$-\frac{\delta}{4}$	$\frac{\delta}{16} - \frac{5}{96}$	$-\frac{5}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_{v2}^v S_{d1}^v$	0	0	$-\frac{\delta}{4}$	$\frac{\delta}{16} - \frac{1}{96}$	$-\frac{5}{192}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_{v3}^v S_{v2}^v$	$-\frac{1}{8}$	0	$\frac{\omega}{8} - \frac{3}{16}$	$-\frac{\omega}{24} + \frac{3}{32}$	$-\frac{\omega}{96} + \frac{3}{64}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
c	$-\frac{1}{8}$	0	$-\frac{1}{4}$	$\frac{7}{96}$	$-\frac{1}{48}$	$-\frac{3}{32}$	$-\frac{1}{12}$	$-\frac{1}{48}$
e	0	0	$-\frac{3}{8}$	$\frac{1}{32}$	$-\frac{3}{64}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$

Table XXIX. Zero-potentials coefficients for the phase diagram shown in Fig. 17.

Lines & points	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
$S_- S_{v3}^v$	0	$-\frac{1}{4}$	$\frac{\omega}{8} + \frac{9}{16}$	$-\frac{\omega}{24} - \frac{7}{48}$	$-\frac{\omega}{96} + \frac{5}{192}$	$\frac{1}{32}$	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_- S_1$	0	$-\frac{1}{8}$	$\frac{\omega}{4} + \frac{1}{4}$	$-\frac{7\omega}{96} - \frac{13}{192}$	$-\frac{\omega}{96} + \frac{5}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_1 S_3$	$-\frac{1}{4}$	0	$\frac{\omega}{4} - \frac{1}{8}$	$-\frac{7\omega}{96} - \frac{1}{192}$	$-\frac{\omega}{96} - \frac{7}{192}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
$S_3 S_{cb}$	0	0	$\frac{\omega}{4} - \frac{13}{8}$	$-\frac{7\omega}{96} + \frac{83}{192}$	$-\frac{\omega}{96} + \frac{5}{192}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_{d1}^v S_{cb}$	0	0	$-\frac{\delta}{4}$	$\frac{\delta}{16} - \frac{7}{96}$	$-\frac{7}{96}$	0	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_{v2}^v S_{d1}^v$	0	0	$-\frac{\delta}{4}$	$\frac{\delta}{16} - \frac{1}{32}$	$-\frac{3}{64}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$
$S_{v3}^v S_{v2}^v$	$-\frac{1}{8}$	0	$\frac{\omega}{8} + \frac{5}{16}$	$-\frac{\omega}{24} - \frac{5}{96}$	$-\frac{\omega}{96} + \frac{5}{192}$	$-\frac{1}{8}$	$-\frac{1}{12}$	$-\frac{1}{12}$
d	0	0	$\frac{5}{8}$	$-\frac{23}{96}$	$-\frac{13}{192}$	$-\frac{1}{64}$	$-\frac{1}{48}$	$-\frac{1}{48}$

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